DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF ARTS- PHILOSOPHY

SEMESTER -IV

MODAL PROPOSITIONAL LOGIC
(ELECTIVE)

ELECTIVE 404

BLOCK-2

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

MODAL PROPOSITIONAL LOGIC

BLOCK-1

BLOCK-2

BLOCK 2: MODAL PROPOSITIONAL LOGIC

Introduction to the Block

Unit 8 deals with Modal logic in the middle Ages. There are four modal paradigms in ancient philosophy: the frequency interpretation of modality, the model of possibility as potency, the model of antecedent necessities and possibilities with respect to a certain moment of time (diachronic modalities)

Unit 9 deals with Varieties of Modality. Modal statements tell us something about what could be or must be the case.

Unit 10 deals with Modal Proposition. The term refers to the argument in which a proposition is arrived at and affirmed or denied on the basis of one or more other propositions accepted as the starting point of the process.

Unit 11 deals with Model Propositional Calculus. As the name suggests, propositional functions are functions that have propositions as their values.

Unit 12 deals with Classical Logic: Some normal prepositional modal system

Unit 13 deals with Modern Origins of Modal Logic: The systems of T.S4 and S5. Modal logic can be viewed broadly as the logic of different sorts of modalities, or modes of truth: alethic ("necessarily"), epistemic ("it is known that"), deontic ("it ought to be the case that"), or temporal ("it has been the case that") among others.

Unit 14 deals with The Lewis system of strict implication. Clarence Irving (C.I.) Lewis was perhaps the most important American academic philosopher active in the 1930s and 1940s.

UNIT 8: MODAL LOGIC IN THE MIDDLE AGES

STRUCTURE

8.0 OBJECTIVES

After this unit, we can able to know:

- Aspects of Ancient Modal Paradigms
- Early Medieval Developments
- Modalities in Thirteenth-Century Logical Treatises
- Fourteenth-Century Discussions

8.1 INTRODUCTION

There are four modal paradigms in ancient philosophy: the frequency interpretation of modality, the model of possibility as a potency, the model of antecedent necessities and possibilities with respect to a certain moment of time (diachronic modalities), and the model of possibility as non-contradictoriness. None of these conceptions, which were well known to early medieval thinkers through the works of Boethius, was based on the idea of modality as involving reference to simultaneous alternatives. This new paradigm was introduced into Western thought in early twelfth-century discussions influenced by Augustine's theological conception of God as acting by choice between alternative histories.

While the new idea of associating modal terms with simultaneous alternatives was used also in thirteenth-century theology, it was not often discussed in philosophical contexts. The increasing acceptance of Aristotle's philosophy in the thirteenth century gave support to traditional modal paradigms, as is seen in Robert Kilwardby's influential commentary on Aristotle's Prior Analytics, in which modal syllogistic is treated as an essentialist theory of the structures of being. There were analogous discussions of philosophical and theological modalities in Arabic philosophy. Arabic modal theories influenced Latin discussions mainly through the translations of Averroes's works.

John Duns Scotus developed the conception of modality as alternativeness into a detailed theory. A logically possible state of affairs is something to which to be is not repugnant, though it may not be compossible with other possibilities. Scotus's modal semantics influenced early fourteenth-century philosophy and theology in many ways. Thirteenth-century essentialist assumptions were dropped from modal syllogistic, the Aristotelian version of which was regarded as a fragmentary theory without a sufficient explication of the various fine structures of modal propositions.

8.2 ASPECTS OF ANCIENT MODAL PARADIGMS

In speaking about the general features of the universe, ancient philosophers were inclined to think that all generic possibilities will be actualized, a habit of thinking called the principle of plenitude by Arthur O. Lovejoy (1936). Correspondingly, it was natural for them to think that the invariant structures of reality are necessary. This line of thought is found, e.g., in Plato's doctrine of the ideas which are exhaustively imitated in the world by the Demiurge, in Aristotle's theory of the priority of actuality over potentiality, in the Stoic doctrine of the rational world-order and the eternal cosmic cycle, as well as in Plotinus' metaphysics of emanation (Knuuttila 1993).

In these approaches to the constituents of the universe, modal notions could be understood in accordance with the so-called 'statistical' or 'temporal frequency' model of modality where the meaning of modal terms is spelled out extensionally as follows: what is necessary is always actual, what is impossible is never actual and what is possible is at least sometimes actual. The term 'statistical interpretation of modality' was introduced into the modern discussion by Oscar Becker (1952), and it has been applied since in descriptions of certain ways of thinking in the history of philosophy as well, particularly by Jaakko Hintikka (1973).

Even though Aristotle did not define modal terms with the help of extensional notions, examples of this habit of thought can be found in his discussion of eternal beings, the natures of things, the types of events, and generic statements about such things. Modal terms refer to the one and only world of ours and classify the types of things and events on the basis of their actuality. This paradigm suggests that actualization is the general criterion of the genuineness of possibilities, but the deterministic implications of this view compelled Aristotle to seek ways of speaking about unrealized singular possibilities. Diodorus Chronus (fl. 300 BCE) was a determinist who found no problem in this way of thinking. Some commentators have argued that Aristotle's views showing similarities to the statistical model are based on special metaphysical and ontological doctrines and not on his conception of modal terms themselves. However, it is not clear that Aristotle had any such distinction in mind. (For different interpretations and evaluations of the role of this model in Aristotle, see Hintikka 1973, Sorabji 1980, Seel 1982, Waterlow 1982a, van Rijen 1989, Gaskin 1995.) In Posterior Analytics I.6 Aristotle states that certain predicates may belong to their subjects at all times without belonging to them necessarily. Some ancient commentators took this to mean that Aristotle operated with a distinction between strong essential per se necessities and weak accidental necessities in the sense of non-essential invariances, such as inseparable accidents (see also Porphyry, Isagoge 3.5–6), and that this distinction played an important role in his modal syllogistic. See the commentaries on the Prior Analytics by Alexander of Aphrodisias (36.25–32; 201.21–24) and Philoponus (43.8–18; 126.7–29); Flannery 1995, 62–65, 99–106. This was also the view of Averroes and some Latin authors in the Middle Ages. (See below.)

Another Aristotelian modal paradigm was that of possibility as potency. In Met. V.12 and IX.1 potency is said to be the principle of motion or change either as the activator or as the receptor of a relevant influence. (For agent and patient in Aristotle's natural philosophy in general, see Waterlow 1982b.) The types of potency-based possibilities belonging to a species are recognized as possibilities because of their actualization — no natural potency type remains eternally frustrated. Aristotle says that when the agent and the patient come together as being capable, the one must act and the other must be acted on (Met. IX.5).

In De Caelo I.12 Aristotle supposes, per impossibile, that a thing has contrary potencies, one of which is always actualized. He argues that the alleged unactualized potencies are not potencies at all because they cannot be assumed to be realized at any time without contradiction. Aristotle applies here the model of possibility as non-contradictoriness which is defined in Prior Analytics I.13 as follows: when a possibility is assumed to be realized, it results in nothing impossible. In speaking about the assumed non-contradictory actualization of a possibility, Aristotle thinks that it is realized in the real history. This argument excludes those potentialities which remain eternally unrealized from the set of genuine possibilities. Aristotle applies here and in some other places (for example, Met IX.4, An. pr. I.15) a reductio argument which consists of a modal inference rule $L(p \rightarrow q) \rightarrow (Mp \rightarrow Mq)$ and the assumption that the possibility is realized (Rosen and Malink 2012; Smith 2016). The argument has created much controversy about how possibilities are supposed to obtain. See Judson 1983; Rini 2011, 135–156.)

Aristotle refers to potencies in criticizing some of his contemporaries who claimed that only that which takes place is possible (Met. IX.3). The model of possibility as potency prima facie allowed him to speak about all kinds of unrealized singular possibilities by referring to passive or active potencies, but taken separately they represent partial possibilities which do not guarantee that their actualization can take place. More is required for a real singular possibility, but when the further requirements are added, such as a contact between the active and passive factor and the absence of an external hindrance, the potency model suggests that the potency can really be actualized only when it is actualized (Met. IX.5, Phys. VIII.1). It is possible that this led Aristotle to define motion (kinêsis) as the actuality of the potentiality (of the end) qua potentiality (Phys. III.1), but this did not explain the possibility of beginning (Hintikka et al. 1977).

In discussing future contingent statements in In Chapter 9 of De interpretatione, Aristotle says that what is necessarily is when it is, but he then qualifies this necessity of the present with the remark that it does not follow that what is actual is necessary without qualification. If he meant that the temporal necessity of a present event does not imply that such an event necessarily takes place in circumstances of that type, this is an unsatisfactory ‘statistical’ attempt to avoid the problem that changeability as a criterion of contingency makes all temporally definite singular events necessary (Hintikka 1973). Another interpretation is that Aristotle wanted to show that the necessity of an event at a certain time does not imply that it would have been antecedently necessary. Aristotle discusses such singular diachronic modalities in some places (Met. VI.3; EN III.5, 1114a17–21; De int. 19a13–17) in which he seems to assume that the conditions which at t1 are necessary for p to obtain at a later time t2 are not necessarily sufficient for this, although they might be sufficient for the possibility (at t1) that p obtains at t2. Aristotle did not elaborate these ideas, which might have been his most promising attempt to formulate a theory of unrealized singular possibilities. (The importance of this model is particularly stressed in Waterlow 1982a; see also von Wright 1984; Weidemann 1986; Gaskin 1995.)

Aristotle’s conceptual difficulties can be seen from his various attempts to characterize the possibilities based on dispositional properties such as heatable, separable, or countable. Analogous discussions were not unusual in later ancient philosophy. In Philo’s definition of possibility (ca. 300 BCE), the existence of a passive potency was regarded as a sufficient ground for speaking about a singular possibility. The Stoics revised this definition by adding the condition of the absence of external hindrance, thinking that otherwise the alleged possibility could not be realized. They did not add that an activator is needed as well, because then the difference between potentiality and actuality would disappear. According to the deterministic world view of the Stoics, fate as a kind of active potency necessitates everything, but they did not accept the Master

Argument of Diodorus Cronus for determinism, which was meant to show that there cannot be possibilities which will not be realized. The number of passive potencies with respect to a definite future instant of time is greater than what will be realized. As long as these possibilities are not prevented from being realized by other things, they in some sense represent open possibilities. Alexander of Aphrodisias thought that it was misleading to speak about about unrealized diachronic possibilities if everything is determined. He argued for what he took to be Aristotle's view, namely that there are undetermined prospective alternatives which remain open options until the moment of time to which they refer. (See Sharples 1983; Bobzien 1993, 1998; Hankinson 1998.) Neither Aristotle nor later ancient thinkers had any considered conception of simultaneous alternatives. They thought that what is necessarily is when it is, and that the alternative possibilities disappear when the future is fixed. Alexander's Peripatetic theory of alternative prospective possibilities could be characterized as the model of diachronic modalities without simultaneous alternatives: there are transient singular alternative possibilities, but those which will not be realized vanish instead of remaining unrealized.

Aristotle often made use of indirect arguments from false or impossible positions by adding hypotheses which he himself labelled as impossible. In order to defend Aristotle's procedure against ancient critics, Alexander of Aphrodisias characterized these hypotheses as impossibilities which were not nonsensical. (For this controversy, see Kukkonen 2002.) Some late ancient authors were interested in impossible hypotheses as tools for conceptual analysis. In the arguments which were called Eudemian procedures something impossible was assumed in order to see what followed. The impossibilities discussed in this way by Philoponus and Boethius show similarities with Porphyry's characterization of inseparable accidents as something which cannot occur separately but can be separated in thought. These hypotheses were not regarded as formulations of possibilities in the sense of what could be actual; they were counterpossible and not merely counterfactual (Martin 1999).

There are several recent works on Aristotle's modal syllogistics, but no generally accepted historical reconstruction which would make it a coherent theory. It was apparently based on various assumptions which were not fully compatible (Hintikka 1973, Smith 1989, Striker 2009). Some commentators have been interested in finding coherent layers of the theory by explicating them in terms of Aristotle's other views (van Rijen 1989; Patterson 1995). There are also several formal reconstructions such as Rini 2011 (modern predicate logic), Ebert and Nortmann 2007 (possible worlds semantics), various set-theoretical approaches discussed in Johnson 2004, and Malink 2006, 2013 (mereological semantics).

8.3 EARLY MEDIEVAL DEVELOPMENTS

Early medieval thinkers were well acquainted with ancient modal conceptions through Boethius' works. One of the Aristotelian modal paradigms occurring in Boethius is that of possibility as potency (potestas, potentia). According to Boethius, when the term 'possibility' (possibilitas) is used in the sense of 'potency', it refers to real powers or tendencies, the ends of which are either actual or non-actual at the moment of utterance. Some potencies are never unrealized. They are said to be necessarily actual. When potencies are not actualized, their ends are said to exist potentially (In Periherm. II.453–455). Necessarily actual potencies leave no room for the potencies of their contraries, for they would remain unrealized forever and the constitution of nature does not include elements which would be in vain (In Periherm. II. 236). The potencies of non-necessary features of being do not exclude contrary potencies. They are not always and universally actualized, but as potency-types even these potencies are taken to satisfy the actualization criterion of genuineness (In Periherm. I.120–1; II.237).

Boethius' view that the types of potencies and potency based possibilities are sometimes actualized is in agreement with the Aristotelian frequency interpretation of modality. This is another Boethian conception of necessity and possibility. He thought that modal notions can be regarded as tools for expressing temporal or generic frequencies. According to the temporal version, what always is is by necessity, and what never is is impossible. Possibility is interpreted as expressing what is at least sometimes actual. Correspondingly, a generic property of a species is possible only if it is exemplified at least in one member of that species (In Periherm. I.120–1, 200–201; II.237).

Like Aristotle, Boethius often treated statement-making utterances as temporally indeterminate sentences. The same sentence can be uttered at different times, and many of these temporally indeterminate sentences may sometimes be true and sometimes false, depending on the circumstances at the moment of utterance. If the state of affairs the actuality of which makes the sentence true is omnitemporally actual, the sentence is true whenever it is uttered. In this case, it is necessarily true. If the state of affairs associated with an assertoric sentence is always non-actual, the sentence is always false and therefore impossible. A sentence is possible only if what is asserted is not always non-actual (I.124-125). Quasi-statistical ideas are also employed in Ammonius' Greek commentary on Aristotle's De interpretatione which shares some sources with Boethius's work (88.12–28) and in Alexander of Aphrodisias' commentary on Aristotle's modal syllogistic. (See Mueller 1999, 23–31.)

In dealing with Chapter 9 of Aristotle's De interpretatione, Boethius argues (II.241) that because

(1) M(pt & ¬ pt)

(1′) It is possible that p obtains at t and not-p obtains at t

is not acceptable, one should also deny

(2) pt & Mt ¬pt

(2′) p obtains at t and it is possible at t that not-p obtains at t.

The denial of (2) is equivalent to

(3) pt → Lt pt

(3′) If p obtains at t then it is necessary at t that p obtains at t.

(2) was generally denied in ancient philosophy and its denial was taken as an axiom by Boethius as well. Correspondingly, (3) shows how the

necessity of the present was understood in ancient thought. Boethius thought that the temporal necessity of p can be qualified by shifting attention from temporally definite cases or statements to their temporally indeterminate counterparts (I.121–122; II.242–243; cf. Ammonius 153.24–26). This was one of Boethius' interpretations of the Aristotelian distinction between necessity now and necessity without qualification. But he also made use of the diachronic model according to which the necessity of p at t does not imply that, before t, it was necessary that p obtains at t.

Boethius developed the diachronic ideas as part of his criticism of Stoic determinism. If it is not true that everything is causally necessitated, there must be genuine alternatives in the course of events. Free choice was the source of contingency in which Boethius was mainly interested, but he thought in addition that according to the Peripatetic doctrine there is a real factor of indeterminacy in the causal nexus of nature. When Boethius refers to chance, free choice, and possibility in this context, his examples include temporalized modal notions which refer to diachronic prospective possibilities at a given moment of time. A temporally determinate prospective possibility may not be realized at the time to which it refers, in which case it ceases to be a possibility. Boethius did not develop the idea of simultaneous alternatives which would remain intact even when diachronic possibilities had vanished, insisting that only what is actual at a certain time is at that time possible at that time (cf. (3) above). But he also thought that there are objective singular contingencies, so that the result of some prospective possibilities is indefinite and uncertain 'not only to us who are ignorant, but to nature' (In Periherm. I.106, 120; II.190–192, 197–198, 203, 207). (For Boethius's modal conceptions, see Kretzmann 1985; Knuuttila 1993, 45–62.)

As for the discussion of future contingent statements in De interpretatione 9, Boethius' view shows similarities to that of Ammonius, both authors apparently having known some similar Greek discussions. (Ammonius's Greek commentary on De interpretatione is translated by D. Blank and Boethius's two Latin commentaries by N. Kretzmann in the same volume, with essays by R. Sorabji, N. Kretzmann and M.

Mignucci, in 1998.) According to the majority interpretation, Ammonius and Boethius ascribe to Aristotle the view that the predictions of future contingent events and their denials differ from other contradictory pairs of propositions, because truth and falsity are not definitely distributed between them and the propositions are consequently neither true nor false. (For various interpretations of how Boethius restricted bivalence, see Frede 1985; Craig 1988; Gaskin 1995, Kretzmann 1998.) Another interpretation holds that future contingents are not definitely true or false in Boethius' view because their truth-makers are not yet determined, but are true or false in an indeterminate way. No qualification of the principle of bivalence is involved (Mignucci 1989, 1998; for related interpretation of Ammonius, see Seel 2000.) While most medieval thinkers regarded the latter view as true, many of them thought that Aristotle's opinion was similar to the majority interpretation of Boethius. Peter Abelard and John Buridan were among those who read Aristotle as holding that future contingent propositions are true or false. Peter Auriol argued that these propositions lack a truth-value; even God is aware of the future in a way which does not imply bivalence. This was an exceptional view. (See Normore 1982, 1993; Lewis 1987; Schabel 2000; Knuuttila 2011.) Boethius, Aquinas, and many others thought that God can know future contingents only because the flux of time is present to divine eternity. Some late medieval thinkers, for example John Duns Scotus and William of Ockham, found the idea of atemporal presence of history to God problematic and tried to find other models for foreknowledge. These discussions led to the so-called middle knowledge theory of the counterfactuals of freedom (Craig 1988; Freddoso 1988; Dekker 2000).

From the point of view of the history of modal thought, interesting things took place in theology in the eleventh and twelfth centuries. Augustine had already criticized the application of the statistical model of possibility to divine power; for him, God has freely chosen the actual world and its providential plan from alternatives which he could have realized but did not will to do (potuit sed noluit). This way of thinking differs from ancient philosophical modal paradigms, because the metaphysical basis is now the eternal domain of simultaneous alternatives instead of the idea of one necessary world order. In Augustine, God's eternal ideas of finite beings represent the possibilities of how the highest being can be imitated, the possibilities thus having an ontological foundation in God's essence. This was the dominating conception of theological modal metaphysics until Duns Scotus departed from it. The discrepancy between the Catholic doctrine of God's freedom and power and the philosophical modal conceptions was brought into the scope of discussion by Peter Damian and Anselm of Canterbury and was developed in a more sophisticated way in twelfth-century considerations of God's power and providence and historical contingencies. While the new idea of associating modal terms with simultaneous alternatives continued to be used in thirteenth-century theology, it was not often discussed in philosophical contexts. The increasing acceptance of Aristotle's philosophy gave support to traditional modal paradigms in logical treatises on modalities, in metaphysical theories of the principles of being, and in the discussions of causes and effects in natural philosophy. (See Holopainen 1996; Knuuttila 2001, 2008; 2012; for Arabic discussions; see also Bäck 2001; Kukkonen 2000, 2002; for divine omnipotence, see Moonan 1994; Gelber 2004, 309–349.) A typical example of the Averroist frequency view of contingency is found in John of Jandun's Questions on Aristotle's De caelo I.34.

In addition to Augustinian theological issues, one can find some theoretical considerations of the new modal semantics in the twelfth century. Even though Abelard made use of traditional modal concepts, he was also interested in the philosophical significance of the idea of modality as alternativeness. Assuming that what is actual is temporally necessary at a certain point of time as no longer avoidable, he adds that unrealized counterfactual alternatives are possible at the same time in the sense that they could have happened at that time. There are also merely imaginable alternatives, such as Socrates' being a bishop, which never had a real basis in things. (See Martin 2001, 2003; Marenbon 2007, 156–158, is sceptical about this interpretation.) Gilbert of Poitiers stressed the idea that natural regularities which are called natural necessities are not absolute, since they are freely chosen by God and can be overridden by divine power. This basically Augustinian conception was a widespread theological view, but in explaining Plato's 'Platonitas' Gilbert argues that this includes all that Plato was, is and will be as well as what he could be but never is (The Commentaries on Boethius 144.77–78, 274.75–76). The modal element of the individual concept was probably needed in order to speak about Plato in alternative possible histories (Knuuttila 1993, 75–82).

An interesting early thirteenth-century philosophical analysis of Augustinian modalities was put forward by Robert Grosseteste (Lewis 1996). Grosseteste taught that while things are primarily called necessary or possible 'from eternity and without beginning' with respect to God's eternal knowledge, there are necessities and impossibilities with a beginning in God's providence which are eternal contingencies in the sense that God could have chosen their opposites (De libero arbitrio 168.26–170.33, 178.24–29). One of the theses of twelfth-century authors, later called nominales, was that 'What is once true is always true'. It was argued that while tensed statements about temporally definite singular events have a changing truth-value, the corresponding non-tensed propositions are unchangingly true or false, without being necessarily true or false for this reason (Nuchelmans 1973, 177–189; Iwakuma and Ebbesen 1992). This was in agreement with Abelard's view that future contingent propositions are true or false. The actuality of a contingent state of affairs at a specified future time does not exclude the non-temporalized possibility of simultaneous alternatives, nor does the truth of a proposition about this state of affairs make it necessary (Glossae super Peri hermeneias IX.520–577; Peter of Poitiers, Sententiae I.7.133–43, I.12.164–223, I.14, 328–353).

8.4 MODALITIES IN THIRTEENTH-CENTURY LOGICAL TREATISES

Modifying Boethius's systematization of Aristotle's remarks in De interpretatione 12 and 13, twelfth- and thirteenth-century logicians often presented the equipollences between modal terms and opposed relations between modal propositions with the help of the following diagram:

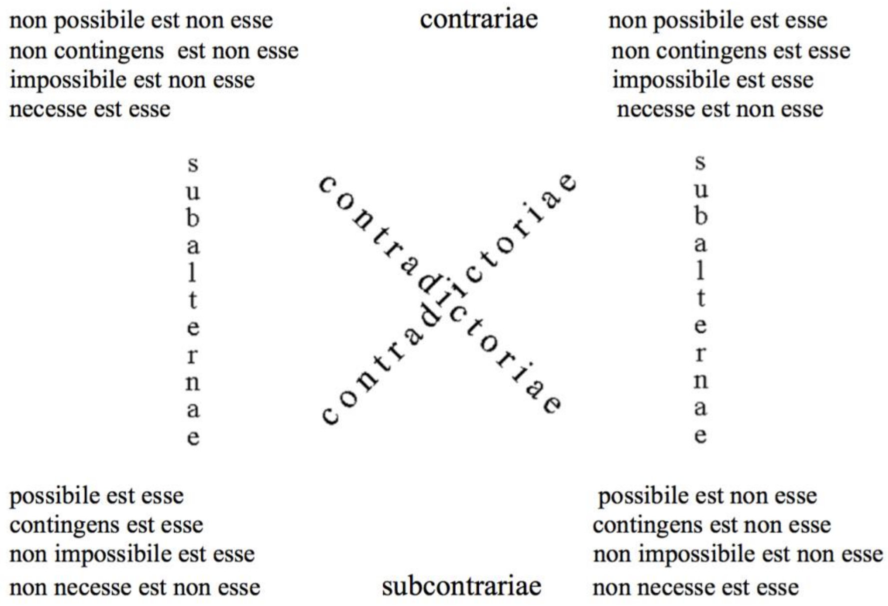

Figure 1.

The square could be taken to refer to modals de dicto or singular modals de re (see below.) Abelard tried to define the opposed relations between quantified de re modals as well, mistakenly thinking that these were the same as those between singular modal propositions (Glossae super Perihermeneias XII.468–471, 530–544). This question was not much discussed before its satisfactory solution in fourteenth-century modal semantics. (See Hughes 1989 and his description of Buridan's octagon of modal opposites and equipollences.)

While possibile and contingens are treated as synonyms in the figure, it became more usual to associate the former with one-sided possibility (not impossible) and the latter with two-sided possibility (neither necessary nor impossible).

The anonymous Dialectica Monacensis (ca. 1200) is one of the numerous works representing the new terminist approach to logic and can be used as an example of how modalities were treated in it. (A collection of late twelfth- and early thirteenth-century logical texts is edited in de Rijk 1962–67.) In discussing the quantity (universal, particular, singular) and quality (affirmative, negative) of the modals, the author states that modal terms may be adverbial or nominal. The modal adverb qualifies the copula, and the structure of the sentence can be described as follows:

(4) quantity/subject/modalized copula/predicate (for example: Some A's are necessarily B)

In this form, the negation can be located in different places, either

(5) quantity/subject/copula modalized by a negated mode/predicate (for example: Some A's are-not-necessarily B)

or

(6) quantity/subject/modalized negative copula/predicate (for example: Some A's are-necessarily-not B)

The modal sentences with nominal modes can be read in two ways. One can apply an adverbial type of reading to them, which is said to be how Aristotle treats modal sentences in the Prior Analytics. The quality and quantity of such a de re modal sentence is determined by the corresponding non-modal sentence. In a de dicto modal sentence that which is asserted in a non-modal sentence is considered as the subject about which the mode is predicated. When modal sentences are understood in this way, they are always singular, their form being:

(7) subject/copula/mode (for example: That some A's are B is necessary.)

This reading is said to be the one which Aristotle presented in De interpretatione (De Rijk 1967, II-2, 479.35–480.26). The idea of the systematic distinction between the readings de dicto (in sensu composito) and de re (in sensu diviso) of modally qualified statements was employed in Abelard's investigations of modal statements (Glossae super Perihermeneias XII, 3–106; Dialectica 191.1–210.19). Independently of Abelard, the distinction was often used, as in the Dialectica Monacensis, in discussions of the composition-division ambiguity of sentences. (See also Maierù 1972, ch. 5; Jacobi 1980, ch. 4.)

The author of the Dialectica Monacensis says that the matter of an assertoric sentence may be natural, remote, or contingent. True affirmative sentences about a natural matter maintain the existence of compounds which cannot be otherwise; these sentences as well as the compounds are called necessary. False affirmative sentences about a remote matter maintain the existence of compounds which are necessarily non-existent; they are called impossible. Sentences about a contingent matter are about compounds which can be actual and which can be non-actual (472.9-473.22). The theory of the modal matter was popular in early medieval logic and was also dealt with in mid-thirteenth-century handbooks. It was sometimes associated with the statistical interpretation of natural modalities, for example by Thomas Aquinas who wrote that universal propositions are false and particular propositions are true in contingent matter (In Periherm. I.13, 168). For the history of modal matter, see Knuuttila 2008, 508–509. Another often discussed theme was the distinction between modalities per se and per accidens which was based on the idea that the modal status of a temporally indefinite sentence may be changeable or not; for example, 'You have not been in Paris' may begin to be impossible, whereas 'You either have or have not been in Paris' may not. (See, for example, William of Sherwood, Introduction to Logic, 41). Another distinction between sentences necessary per se and per accidens was based on Aristotle's theory of per se predication in Posterior Analytics I.4. A sentence was said to be accidentally necessary when it was unchangeably true but, as distinct from per se predications, there was no necessary conceptual connection between subject and predicate. This became an important part of thirteenth-century interpretations of Aristotle's modal syllogistics. (See, for example, Robert Kilwardby's Notule libri Priorum 8.133–142; 40.162–174.)

One example of the prevalence of the traditional use of modal notions can be found in the early medieval de dicto/de re analysis of examples such as 'A standing man can sit'. It was commonly stated that the composite (de dicto) sense is 'It is possible that a man sits and stands at the same time' and that on this reading the sentence is false. The divided (de re) sense is 'A man who is now standing can sit' and on this reading the sentence is true. Many authors formulated the divided possibility as follows: 'A standing man can sit at another time'. It was assumed that a possibility refers to an actualization in the one and only world history and that it cannot refer to the present moment because of the necessity of the present understood in the Aristotelian sense formulated in (2) and (3) above. When authors referred to another time, they thought that the possibility will be realized at that time or that the divided possibility refers to the future even though it may remain unrealized. Those who made use of the (at that time modern) idea of simultaneous alternatives took the composite reading to refer to one and the same state of affairs and the divided reading to simultaneous alternative states of affairs. This analysis was also applied to the question of whether God's knowledge of things makes them necessary (Knuuttila 1993, 118–121).

A great deal of Abelard's logical works consisted of discussions of topics, consequences and conditionals. Like Boethius, Abelard thought that true conditionals express necessary connections between the antecedents and the consequents. Abelard argued that inseparability and entailment between the truth of the antecendent and consequent are required for the truth of a conditional. Some twelfth-century masters regarded the principle that the antecedent is not true without the consequent as a sufficient condition for the truth of a conditional and accepted the so-called paradoxes of implication. The question of the nature of conditionals and consequences remained a popular theme in medieval logic (Martin 1987, 2012).

The principles of propositional modal logic, found in Prior Analytics I.15, were generally expressed as follows: if the antecedent of a valid consequence is possible/necessary, the consequent is possible/necessary (Abelard, Dialectica 202.6–8). However, the main interest was in modal syllogistic and modal predicate logic. Avicenna (d. 1037) wrote a brief Arabic summary of Aristotle's modal syllogistic, but his own theory was different, being based on the assumptions that the subject terms and the predicate terms of assertoric and modal propositions stand for all possible applications and the truth-conditions of assertoric propositions and corresponding possibility propositions are the same. It follows, for example, that syllogisms with assertoric premises coincide with uniform possibility syllogisms (Street 2002, 2005). Avicenna was particularly interested in relative necessities and distinguished between various types of conditional necessities in terms of temporal determinations. Later Arabic works on modal theories were much influenced by Avicenna. (See Strobino and Thom 2016.) While Averroes's commentaries on the Prior Analytics followed the main lines of Aristotle's text, his separate treatise on modality involved new systematic ideas, mainly the theory of accidental and per se necessary terms and the interpretation of syllogistic necessity premises as per se necessary predications with per se necessary terms. Both ideas were inspired by Aristotle's remarks in the Posterior Analytics I.4; Averroes's syllogistic applications were probably influenced by ancient sources. Since Averroes took modal premises to be of the divided type, assertoric premises in Aristotelian mixed necessity-assertoric syllogisms must have a predicate term which is necessary. The same applies to the subject term of the first premise in mixed assertoric-necessity syllogisms (Quaesita octo in librum Priorum Analyticorum, IV.3, 84, in Aristotelis Opera cum Averrois Commentariis I.2b; see also Thom 2003, 81–85). This is a speculative explanation of Aristotle's asymmetric treatment of mixed necessity-assertoric syllogisms and mixed assertoric-necessity syllogisms. Gersonides later tried to develop further Averroes's remarks; see Manekin 1992. Analogous essentialist ideas were developed in thirteenth-century Latin discussions.

The first Latin commentary on the Prior Analytics is an anonymous late twelfth-century treatise ('Anonymus Aurelianensis III') which involves detailed discussions of modal conversion and modal syllogisms as well as many problems dealt with in ancient commentaries. (See Ebbesen 2008; an edition by Thomsen Thörnqvist 2015; see also Bydén and Thomsen Thörnqvist, eds., 2017). Dialectica Monacensis involves a brief summary of Aristotle's modal syllogistic the elements of which were discussed in logic courses in Paris in the first part of the thirteenth century. Robert Kilwardby's commentary Notule libri Priorum (c. 1240) became an authoritative work from which the discussions of modal syllogisms in the commentaries of Albert the Great (ca. 1250) and many others were largely derived (Knuuttila 2008, 545–548). Abelard, who did not deal with Aristotle's modal syllogistic, said that the modals in mixed syllogisms with both modal and assertoric premises should be understood in a way which he elsewhere characterizes as de re interpretation (Glossae super Perihermeneias XII.189–203). This reading of modal premises was often assumed, although it was seldom discussed as such. A central problem of Aristotle's theory is that the structure of the premises is not analyzed. Even if it is natural to think that the presupposition of the mixed moods is a de re reading of modally qualified premises, this creates difficulties when applied to the conversion rules, most of which are unproblematic only if understood as rules for modals de dicto. (According to Aristotle, necessity premises are converted in the same way as assertoric premises, 'Every/some A is B' implies 'Some B is A' and 'No A is B' implies 'No B is A'. Negative contingency premises are converted to corresponding affirmative contingency propositions and these by the conversion of terms to particular contingency propositions.)

While many historians think that Aristotle's modal syllogistic included incompatible elements, this was not the view of mid-thirteenth century logicians. Many of them discussed the same alleged counter-examples to the universal convertibility of necessary propositions, such as

(8) Everything healthy (or awake) is necessarily an animal.

Robert Kilwardby's explanation is based on the view that convertible necessity premises are necessity propositions per se and not per accidens, like (8), which are not convertible. (See Notule libri Priorum 8.133–146.) In affirmative necessity propositions per se, the subject is per se connected to the predicate. In negative necessity propositions per se, the subject is per se incompatible with the predicate. The terms in per se inherences or incompatibilities are essential and necessarily stand for the things they signify. The historical background of Kilwardby's interpretation is not clear, but it does show similarities to Averroes's discussion mentioned above. (See Lagerlund 2000, 25–42; Thom 2007, 19–28.)

As for the conversion of contingency propositions (neither necessary nor impossible), Kilwardby notes that while the converted propositions of indefinite (utrumlibet) contingency are of the same type of contingency, the conversion of natural contingency propositions (true about most cases) results in contingency propositions when contingency means possibility proper (not impossible). There were extensive discussions of the kinds of contingency based on various philosophical ideas in Kilwardby, Albert the Great and their contemporaries (Knuuttila 2008, 540–541).

Following Aristotle's remark that 'A contingently belongs to B' may mean either 'A contingently belongs to that to which B belongs' or 'A contingently belongs to that to which B contingently belongs', Kilwardby argues that the subject terms in contingency syllogisms are read in the second way and ampliated, if syllogistic relations do not demand restrictions. In explaining the difference in this respect between necessity propositions and contingency propositions, he states that since the terms in per se necessity propositions are essential, ‘Every A is necessarily B’ and ‘Whatever is necessarily A is necessarily B’ behave in the same way in logic. Contingency propositions which are ampliated do not mean the same as those which are not ampliated (Notule libri Priorum 18.187–207; 18.653–672).

According to Kilwardby, the modal character of the predication in the conclusion of the perfect first figure syllogism follows from the first premise, which involves the whole syllogism in accordance with the dici de omni et nullo (Lagerlund 2000, 41–42). The premises and the conclusion in uniform necessity syllogisms are necessary per se. In mixed first-figure syllogisms with a major necessity premise and a minor asertoric premise, the non-modalized premise should be simpliciter assertoric, i.e., a necessarily true per se predication. Similarly, in mixed first-figure syllogisms with contingent major and assertoric minor premises, the assertoric premise must be simpliciter assertoric, but this time the criteria are that the predicate belongs to the subject per se, invariably or by natural contingency (Notule libri Priorum 15.255–301; 20.706–736).

Kilwardby explains that in first-figure mixed necessity-assertoric syllogisms the necessity premise appropriates to itself a minor which is necessary per se; no such appropriation occurs in first-figure mixed assertoric-necessity syllogisms. There are similar appropriation rules for some mixed second-figure and third-figure moods with assertoric and necessity premises and for various mixed contingency moods pertaining to the kind of appropriated contingency premises or assertoric premises (Thom 2007, chs. 5–6).

Kilwardby and his followers regarded Aristotle’s modal syllogistic as the correct theory of modalities, the explication of which demanded various metaphysical considerations. As exemplified by the appropriation rules, they assumed that propositions of the same form had different interpretations, depending on how they were related to other propositions in a syllogism. From the logical point of view, these rules have an ad hoc character. (For some comparisons between contemporary philosophical modal logic and thirteenth-century views, see also Uckelman 2009.)

After Kilwardby and Albert, several thirteenth-century authors wrote treatises on the Prior Analytics. These are not yet edited; the next edited text is Richard Campsall's early fourteenth-century Questions on Aristotle's Prior Analytics. It shows which kind of questions were found relevant in the tradition influenced by Kilwardby's commentary. Campsall thinks that one should discuss de dicto and de re modalities separately. He says that an affirmative de re possibility statement as of now implies the corresponding assertoric statement (5.40) and a negative assertoric statement as of now implies the corresponding de re necessity statement (5.50). It follows that what is possible now is actualized and things cannot be otherwise because all true present tense negative statements are necessarily true. This is Campsall's version of the traditional doctrine of the necessity of the present. When he says that an affirmative assertoric statement does not imply the corresponding de re necessity statement, the background of this remark is the definition of a de re contingency statement as a conjunction of an affirmative and corresponding negative possibility proper statement (7.34–36). For the same reason, a negative de re possibility statement does not imply the corresponding assertoric statement. Campsall equates de re necessity with respect to actual things to unchanging predication and contingency to changing predication. Actual things may be contingent in the sense that they will be changed in the future (12.31). For a different interpretation of Campshall's confusing formulations, see Lagerlund 2000, 87–90).

8.5 FOURTEENTH-CENTURY DISCUSSIONS

John Duns Scotus's modal theory can be regarded as the first systematic exposition of the new intensional theory of modality, some elements of which were put forward in the twelfth century. In criticizing Henry of Ghent's theory of theological modalities, Scotus sketched the famous model of 'divine psychology' in which certain relations between theological, metaphysical, and modal notions are defined. Scotus deviated from the metaphysical tradition in which possibilities were founded in divine being. According to him, when God as an omniscient being knows all possibilities, he does not know them by turning first to his essence. Possibilities can be known in themselves (Ord. I.35, 32). In fact they would be what they are even if there were no God. Scotus states that if it is assumed that, per impossibile, neither God nor the world exists and the proposition 'The world is possible' then existed, this proposition would be true. The actual world is possible as it is, and this possibility and the possibilities of unrealized things are primary metaphysical facts which are not dependent on anything else (Ord. I.7.1, 27; Lect. I.7, 32, I.39.1–5, 49).

Scotus calls the propositional formulations of pure possibilities 'logical possibilities'. These express things and states of affairs to which it is not repugnant to be. Possibilities as such have no kind of existence of their own nor are they causally sufficient for the existence of anything, but they are real in the sense that they form the precondition for everything that is or can be. God's omniscience involves all possibilities and as object of divine knowledge they receive an intelligible or objective being. Some of these are included in God's providential plan of creation and will receive an actual being. The description of how things could be at a certain moment consists of compossible possibilities. Though possibilities necessarily are what they are, the actualization of non-necessary possibilities is contingent. Since all finite things are contingently actual, their alternatives are possible with respect to the same time, though these are not compossible with what is actual. Impossibilities are incompossibilities between possibilities (Ord. I.35, 32, 49–51, I.38, 10, I.43, 14; Lect. I.39.1–5, 62–65).

In criticizing extensional modal theories Scotus redefined a contingent event as follows: 'I do not call something contingent because it is not always or necessarily the case, but because its opposite could be actual at the very moment when it occurs' (Ord. I.2.1.1–2, 86). This is a denial of the traditional thesis of the necessity of the present and the temporal frequency characterization of contingency. In Scotus's modal semantics, the meaning of the notion of contingency is spelt out by considering simultaneous alternatives. What is actual is contingently so if, instead of being actual, it could be not actual. This conception of simultaneous contingent alternatives is part of an argument that the first cause does not act necessarily. According to Scotus, the eternal creative act of divine will is free only if it could be other than it is in a real sense (Lect. I.39.1–5, 58). (For Scotus's modal theory, see Vos et al. 1994; Knuuttila 1996; King 2001; Normore 2003; Hoffmann 2009.)

Scotus's approach to modalities brought new themes into philosophical discussion. One of these was the idea of possibility as a non-existent precondition of all being and thinking. Some of his followers and critics argued that if there were no God, there would not be any kind of modality (see Hoffmann 2002, Coombs 2004; for Bradwardine's criticism, see Frost 2014). Scotus's views were known in the seventeenth century through the works of Suárez and some Scotist authors (Honnefelder 1990). In his discussion of eternal truths, Descartes criticized the classical view of the ontological foundation of modality as well as the Scotist theory of modality and conceivability. (There are different interpretations of Descartes's view of the foundations of modality and how it is related to late medieval discussions; see Alanen 1990; Normore 1991, 2006.)

Another influential idea was the distinction between logical and natural necessities and possibilities. In Scotus's theory, logically necessary attributes and relations are attached to things in all those sets of compossibilities in which they occur. Against this background one could ask which of the natural invariances treated as necessities in earlier natural philosophy were necessary in this strong sense of necessity, and which of them were merely empirical generalizations without being logically necessary. (For a discussion of logical and natural necessities in the fourteenth century, see Knuuttila 1993, 155–160, 2001a.) Buridan applied the frequency model in natural philosophy, and it was often used in early modern thought as well (Knebel 2003).

One important branch of medieval logic developed in treatises called De obligationibus dealt, roughly speaking, with how an increasing set of true and false propositions might remain coherent. According to thirteenth-century rules, a false present tense statement could be accepted as a starting point only if it was taken to refer to a moment of time different from the actual one. Scotus deleted this rule, based on the Aristotelian axiom of the necessity of the present, and later theories accepted the Scotist revision. In this new form, obligations logic could be regarded as a theory of how to describe possible states of affairs and their mutual relationships. These discussions influenced the philosophical theory of counterfactual conditionals (Yrjönsuuri 1994, 2001; Gelber 2004; Dutilh Novaes 2007).

In dealing with counterfactual hypotheses of indirect proofs mentioned above, Averroes and Thomas Aquinas made use of the idea of abstract possibilities which did not imply the idea of alternative domains. The possibilities of a thing can be dealt with at various levels which correspond to Porphyrian predicables. Something which is possible for a thing as a member of a genus can be impossible for it as a member of a species. The same holds of it as a member of a species and an individuated thing. Thus humans can fly because there are other animals which can fly. These abstract possibilities are impossible in the sense that they cannot be actualized. Buridan heavily criticized this approach from the point of view of his new modal theory. He argued that if a counterfactual state of affairs is possible, it can be coherently imagined as actual. If something cannot be treated in this way, calling it possible is based on a conceptual confusion. (See Knuuttila and Kukkonen 2011.) While Scotus, Buridan and many others understood the basic level of possibility in terms semantic consistency, Ockham wanted to preserve the link to the notion of power in his modal considerations, thinking that necessity is actuality plus immutability, the past and the present are necessary, and Scotus was wrong in assuming that things could be different from how they are at the very moment of their actuality (Normore 2016).

Influenced by the new philosophical ideas about modality, William of Ockham (Summa logicae), John Buridan (Tractatus de consequentiis, Summulae de Dialectica) and some other fourteenth-century authors could formulate the principles of modal logic much more completely and satisfactorily than did their predecessors. Questions of modal logic were discussed separately with respect to modal propositions de dicto and de re; modal propositions de re were further divided into two groups depending on whether the subject terms refer to actual or possible beings. It was thought that logicians should also analyze the relationships between these readings and, furthermore, the consequences having various types of modal sentences as their parts. Ockham, Buridan and their followers largely dropped thirteenth-century essentialist assumptions from modal syllogistic. They regarded the Aristotelian version as a fragmentary theory in which the distinctions between different types of fine structures were not explicated and, consequently, did not try to reconstruct Aristotle's modal syllogistic as a consistent whole by one unifying analysis of modal propositions; they believed, like some modern commentators, that such a reconstruction was not possible. (For fourteenth-century modal logic, see King 1985; Lagerlund 2000; Thom 2003; Knuuttila 2008, 551–567.)

According to Hughes (1989), one could supply a Kripke-style possible worlds semantics to Buridan's modal system. Comparing Buridan's general ideas with this may be of heuristic value, although many theoretical questions of modern formal semantics were not those of medieval logicians. (See also Klima 2001.) Ockham and Buridan state that the truth of 'A white thing can be black' demands the truth of 'This can be black' and that 'This can be black' and ''This is black' is possible' mean the same. Compound (de dicto) and divided (de re) readings do not differ at this level, but are separated in dealing with universal and particular propositions. While Ockham did not discuss unrestricted divided necessity propositions, Buridan took the subject terms of all quantified divided modal propositions as standing for possible beings if they are not restricted. The truth of these propositions demands the truth of all or some relevant singular propositions of the type just mentioned; the demonstrative pronoun is then taken to refer to the possible beings even though they may not exist. Buridan could have said that the possible truth of 'This is X' means that it is true in a possible state of affairs in which the possible being referred to by 'this' occurs and that the necessary truth of 'This is X' means that it is true in all possible states of affairs in which the possible being referred to by 'this' occurs.

Notes

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. Discuss the Aspects of Ancient Modal Paradigms.

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2. What do you mean by Early Medieval Developments?

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3. Discuss the Modalities in Thirteenth-Century Logical Treatises.

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4. Discuss the Fourteenth-Century Discussions.

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8.6 LET US SUM UP

The new modal logic was among the most remarkable achievements of medieval logic. Buridan's modal logic was dominant in late medieval times, being more systematic than that of Ockham because of its symmetric treatment of possibility and necessity. It was embraced by Marsilius of Inghen, Albert of Saxony, Jodocus Trutfetter and others (Lagerlund 2000, 184–227; for the later influence of medieval modal theories, see also Coombs 2003; Knebel 2003; Roncaglia 1996, 2003; Schmutz 2006). The rise of the new modal logic was accompanied by elaborated theories of epistemic logic (Boh 1993) and deontic logic (Knuuttila and Hallamaa 1995).

8.7 KEY WORDS

Modal Logic: Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. A modal—a word that expresses a modality—qualifies a statement

8.8 QUESTIONS FOR REVIEW

1. Discuss the concept of Modal Logic in Middle age.
2. Write in details about the concept of Modal Logic.

8.9 SUGGESTED READINGS AND REFERENCES

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8.10 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. See Section 8.2
2. See Section 8.3
3. See Section 8.4
4. See Section 8.5

UNIT 9: VARIETIES OF MODALITY

STRUCTURE

9.0 OBJECTIVES

After this unit, we can able to know:

- To know about the Epistemic and Metaphysical Modality
- To discuss about Metaphysical and Nomic Modality
- To understand the Structure of the Modal Realm

9.1 INTRODUCTION

Modal statements tell us something about what could be or must be the case. Such claims can come in many forms. Consider:

No one can be both a bachelor and married. ('Bachelor' means 'unmarried man'.)

You could not have been born of different parents. (Someone born of different parents wouldn't be you.)

Nothing can travel faster than light. (It's a law of nature.)

One cannot get from London to New York in less than one hour. (Planes that fast haven't been developed yet.)

You cannot leave the palace. (The doors are locked.)

You cannot promise to come and then stay at home. (It's just wrong.)

You cannot start a job application cover letter with “hey guys”. (It’s just not done.)

You cannot castle if your king is in check. (It’s against the rules.)

You cannot deduct your holidays from your taxes. (It’s against the law.)

Fred cannot be the killer. (The evidence shows that he’s innocent.)

Each of these claims appears to have a true reading. But it also seems that ‘cannot’ needs to be interpreted in different ways to make the different sentences true. For one thing, we can, in the same breath, accept a modal claim in one of the senses illustrated by (1)–(10) while rejecting it in another one of these senses, as in the following dialogue:

Caesar: You’re lucky that I’m still here. The doors were unlocked. I could have left the palace.

Cleopatra: True. But then again, you couldn’t have left the palace. That would have been wrong, given that you promised to meet me here.

Moreover, the modal claims (1)–(10) appear to be true for completely different reasons. For example, it may be held that the truth of (1) is due to the meanings of its constituent expressions; that (2) holds because it lies in your nature to be born of your actual parents; that (3) is true because the laws of nature preclude superluminal motion; that (4) holds because of technological limitations; that (5) owes its truth to the presence of insurmountable practical obstacles; that (6)–(9) are made true by the demands of morality, etiquette, the rules of chess, and the law respectively; and that (10) holds because the known facts prove Fred’s innocence.

It is one of the tasks of a philosophical theory of modality to give a systematic and unified account of this multiplicity of modal concepts. This article discusses a few of the main issues that need to be addressed by anyone pursuing this goal. Sections 1 and 2 concern the question of what fundamental categories of modal notions there are. The focus will be on two contemporary debates: whether there are separate forms of modality that are tied to the epistemic and the metaphysical domains (section 1), and whether there is a special kind of necessity associated with the laws of nature (section 2). Section 3 discusses questions about the relations between different notions of necessity. Can some of them be reduced to other, more fundamental ones? If so, which concepts of necessity are the most fundamental ones? And if there are several fundamental kinds of necessity, what do they have in common that makes them all kinds of necessity?

9.2 EPISTEMIC AND METAPHYSICAL MODALITY

There are many ways the world could have been. You could have gotten up later today. Your parents could have failed to meet, so that you were never born. Life could never have developed on earth. The history of the universe could even have been completely different from the beginning. And many philosophers believe that the laws of nature could have been different as well (although that has been denied, as discussed in section 2). Maximally specific ways the world could have been are commonly called 'possible worlds.' The apparatus of possible worlds allows us to introduce a set of modal notions: a proposition is necessary just in case it is true in all possible worlds, a proposition is possible just in case it is true in some possible worlds, and it is contingent just in case it is true in some but not all possible worlds. A sentence is necessary (possible, contingent) just in case it expresses a necessary (possible, contingent) proposition.

The modal notions considered in the last paragraph are not obviously epistemological. On the face it, we are not reporting a fact about what is or can be known or believed by anyone when we say that life could have failed to develop. But there is also a family of modal concepts that are clearly epistemological. These are the notions we employ when we say things like 'Fred must have stolen the book (the evidence shows conclusively that he did it),' or 'Mary cannot be in London (she would have called me).' These modal utterances seem to make claims about what the available evidence shows, or about which scenarios can be ruled out on the basis of the evidence. More formally, we can say that a proposition PP is epistemically necessary for an agent A just in case the empirical evidence AA possesses and ideal reasoning (i.e., reasoning unrestricted by cognitive limitations) are sufficient to rule out ∼P∼P. This notion of epistemic necessity is agent-relative: one and the same claim can be epistemically necessary for one agent, but not for another agent with less empirical evidence. We obtain a notion of epistemic necessity of particular philosophical interest by focusing on a limiting case, namely that of a possible agent with no empirical evidence whatsoever. A proposition PP is epistemically necessary for such an agent just in case ideal reasoning alone, unaided by empirical evidence, is sufficient to rule out ∼P∼P. A proposition that meets this condition can be called a priori in at least one sense of this term, or we can call it simply epistemically necessary (without relativization to an agent). Propositions that are not a priori are called a posteriori.

It is an important and controversial question whether the necessary propositions are all and only the epistemically necessary (a priori) ones, or whether the extensions of the two concepts can come apart. One possible reason for thinking that the notions are coextensive derives from a very natural picture of information and inquiry. On this picture, all information about the world is information about which of all possible worlds is realized (i.e., about where in the space of all possible worlds the actual world is located). My total information about the world can be identified with the set of possible worlds that I cannot rule out on the basis of my empirical evidence and ideal reasoning. As I gather more and more empirical evidence, I can progressively narrow down the range of possibilities. Suppose, for example, that I am ignorant of the current weather conditions. The worlds compatible with my evidence include some where the weather is good and others where it is bad. A look out of the window at the rain provides information about the matter. I can now narrow down the set of possibilities by excluding all possible worlds with fine weather. On this account, a proposition PP is epistemically necessary for AA just in case PP is true in all possible worlds that cannot be ruled out on the basis of AA’s empirical evidence and ideal reasoning. PP is a priori just in case it is epistemically necessary for a possible agent who has no empirical evidence. Since such an agent cannot rule out any possible worlds, a proposition is a priori just in case it is true in all possible worlds. In other words, the a priori propositions are all and only the necessary propositions.

This approach is often combined with a certain account of semantic content. One of the main purposes of language is to transmit information about the world. Where PP is any sentence used for that purpose (roughly speaking, a declarative sentence), it seems natural to think of PP's content (the proposition expressed by it) as the information that is semantically encoded in it. Combining this with the foregoing account of information, we can think of the content of a sentence as a set of possible worlds (namely, the set containing just those worlds of which the sentence is true) or, equivalently, as a function from worlds to truth-values.

This picture connects the modal, epistemic and semantic realms in a simple and elegant way, and various versions of it have informed the work of numerous contemporary philosophers (including David Lewis, Robert Stalnaker, David Chalmers, and Frank Jackson). However, the approach has come under pressure from data to be considered in the next section.

9.2.1 The Data

The idea that all and only the a priori truths are necessary was thrown into serious doubt by the work of philosophers including Hilary Putnam (1972) and Saul Kripke (1980). Kripke distinguishes between two different kinds of singular terms, rigid and non-rigid ones. A so-called rigid designator is an expression that singles out the same thing in all possible worlds. Kripke argues that ordinary proper names like 'Al Gore' are rigid. We can use this name to describe how things actually are, e.g., by saying 'Al Gore became vice president in 1993.' In such cases, the name picks out Al Gore. But we can equally use the name to describe how things stand in other possible worlds, e.g., by saying, 'If Bill Clinton had chosen a different running mate, Al Gore would not have become vice president.' In this case, we are talking about a non-actualized possibility, and we use the name 'Al Gore' to describe this possibility. Moreover, we use the name to say something about how things stand with Al Gore in that possibility. In general, when we use the name to describe any possible world, we use it to talk about the same person, Al Gore. Other examples of rigid designators include indexical expressions like the first-person pronoun 'I,' or the expression 'now.' When you use the term 'I' to describe any possible world, you are always picking out the same thing: yourself. Natural kind terms like ‘water’ and ‘gold’ can also be regarded as rigid terms, as they single out the same kinds in every possible world. Non-rigid singular term, by contrast, pick out different entities in different possible scenarios. The paradigmatic examples of non-rigid terms are descriptions that are satisfied by different objects in different possible worlds. For example, ‘the most annoying person in the history of the world’ may pick out Fred in the actual world, while picking out Cleopatra in some other possible worlds.

Singular terms can be introduced into the language with the help of descriptions. There are two ways in which that can be done. On the one hand, we can stipulate that the singular term is to be synonymous with the description, for example by laying down that ‘the morning star’ is to mean the same as ‘the last celestial body to be seen in the morning.’ When we use the expression to describe another possible world, the new expression will single out whatever celestial body is the last one that can be seen in the morning in that world. Since different things meet this condition in different worlds, the expression is non-rigid. On the other hand, we may introduce a term with the stipulation that it is to be a rigid designator referring to whatever object actually satisfies the description. For instance, we may lay it down that ‘Phosphorus’ is to refer rigidly to the object that is actually the last celestial body visible in the morning. Since that object is Venus, the name will pick out Venus, not only when we use it to describe the actual world, but also when we (in the actual world) use it to describe other possible worlds, including worlds where Venus is not the last planet visible in the morning.When a description is used to introduce a singular term in the second way, it merely serves to fix the reference of the term, but is not synonymous with it.

Now consider a true identity statement that involves two rigid designators, such as

- (1)Mark Twain (if he exists) is Samuel Clemens.

Since ‘Mark Twain’ and ‘Samuel Clemens’ pick out the same entity in every possible world where they pick out anything, this identity statement is a necessary truth. (Note that the statement is conditionalized on Mark Twain’s existence, which makes it possible to avoid the question whether (1) is true in worlds where the two names pick out nothing.) But it is far from immediately obvious that (1) expresses something that can be known a priori. At least on the face of it, we may think that someone who knows her neighbor by the name of 'Samuel Clemens,' who has read several stories by an author named 'Mark Twain' and who fails to realize that her neighbor and the author are identical may not know that which is expressed by (1). Moreover, it may seem that her ignorance is irremediable by reasoning alone, that she requires empirical evidence to come to know that which is stated by (1).

Another type of apparent counterexample to the thesis that all and only the a priori truths are necessary concerns sentences like

- (2)If gold exists, then it has atomic number 79.

It seems plausible that it is an essential property of gold to have atomic number 79: gold could not have (existed but) failed to have that property. (A substance in another possible world that fails to have atomic number 79 simply isn't gold, no matter how similar it may otherwise be to the gold of the actual world.) And yet it seems clear that it can only be known empirically that gold has that atomic number. So, while (2) is a necessary truth, what it says cannot be known a priori. For another illustration of this phenomenon, suppose that I point to the wooden desk in my office and say:

- (3)If this desk exists, it is made of wood.

It is arguably essential to this desk to be made of wood. A desk in another possible world that isn't wooden simply can't be this desk, no matter how similar it may otherwise be to my desk. But it seems that we need empirical evidence to know that the desk is made of wood. So, (3) is another apparent example of a necessary a posteriori truth.

Just as Kripke claims that some truths are necessary without being a priori, he argues that a truth can be a priori without being necessary. To use an example of Gareth Evans's (1982), suppose that I introduce the term 'Julius' by stipulating that it is to refer rigidly to the person who is in fact the inventor of the zip (if such a person exists). Then it may appear that I don't need further empirical evidence to know that

- (4)If Julius exists, then Julius is the inventor of the zip.

But (4) does not seem to be a necessary truth. After all, Julius could have become a salesperson rather than an inventor.

According to Kripke, our initial surprise at the divergent extensions of a prioricity and necessity should be mitigated on reflection. A prioricity (epistemic necessity) is an epistemological notion: it has to do with what can be known. That is not true of the concept of necessity. (2) is necessary because the atomic number of gold is an essential feature of it, and on the face of it, that has nothing to do with what is known or believed by anyone. This kind of necessity is a metaphysical notion, and we may use the term 'metaphysical necessity' to distinguish it more clearly from epistemic necessity.

Kripke's examples are not the only ones that could be appealed to in order to shed doubt on the coextensiveness of necessity and a prioricity. Some other problematic cases are listed below (Chalmers 2002a; cp. Chalmers 2012, ch. 6).

i. Mathematical truths. It is common to hold that all mathematical truths are necessary. But on the face of it, there is no guarantee that all mathematical truths are knowable a priori (or knowable in any way at all). For example, either the continuum hypothesis or its negation is true, and whichever of these claims is true is also necessary. But for all we know, there is no way for us to know that that proposition is true.

ii. Laws of nature. Some necessitarians about the natural laws (see section 2) believe that the laws hold in all metaphysically possible worlds. But they are not a priori truths.

iii. Metaphysical principles. It is often believed that many metaphysical theses are necessary if true, e.g., theses about the nature of properties (e.g., about whether they are universals, sets or tropes) or ontological principles like the principle of unrestricted mereological composition (which says that for any things there is something that is their sum). But it is not obvious that all truths of this kind are a priori. (For discussion, see Chalmers 2012, §§6.4–6.5; Schaffer forthcoming.)

iv. Principles linking the physical and the mental. Some philosophers hold that all truths about the mental are metaphysically necessitated by the physical truths, but deny that it is possible to derive the mental truths from the physical ones by a priori reasoning (see Hill & McLaughlin 1999; Yablo 1999; Loar 1999; and Chalmers 1999 for discussion). On that account, some of the conditionals that link physical and mental claims are metaphysically necessary but not a priori.

These examples are controversial. For any given mathematical claim whose truth-value is unknown, one could hold that it is only our cognitive limitations that have prevented us from establishing or refuting the statement, and that the question could be decided by ideal reasoning (so that the truth of the matter is a priori). Alternatively, it may be held that the truth-value of the mathematical statement is indeterminate. (Perhaps our practices do not completely determine the references of all the terms used in the mathematical claim). The same two options are available in the case of metaphysical principles. Alternatively, one may argue that the relevant metaphysical theses are merely contingent (see, e.g., Cameron 2007). Necessitarianism about the natural laws is highly controversial and may simply be denied. And in response to (iv), one may deny that the physical truths metaphysically necessitate the mental truths (Chalmers 1996), or one may hold that the mental truths can be derived from the physical ones by a priori reasoning (Jackson 1998).

Philosophers have paid more attention to the examples given by Kripke than to other possible cases of the necessary a posteriori, and for that reason the discussion in the rest of this section will mostly focus on Kripke's cases. Two strategies for explaining these examples can be distinguished. Dualists about metaphysical and epistemic modality (dualists, for short) hold that the phenomena reflect a deep and fundamental distinction between two kinds of modality. Monists, by contrast, believe that all the data can ultimately be explained by appeal to a single kind of modality. They may agree that there are cases in which a single sentence is, in some sense, both necessary and a posteriori, or both contingent and a priori. But they insist that there is no similar distinction at the level of worlds or propositions. Rather, the phenomenon arises because a single sentence can be associated with two different propositions, one that is necessary and another that is contingent.

9.2.2 Dualism

Notes

Dualists distinguish between two concepts of propositional necessity, metaphysically necessity and epistemic necessity. The two notions are not coextensive. At least some of the sentences in Kripke's examples express propositions that possess the one kind of necessity but not the other.[4]

Once the existence of a distinctively metaphysical form of propositional necessity is accepted, it is natural to wonder whether it is possible to say more about its nature. Kit Fine (1994) offers an account of it that appeals to the traditional distinction between those properties of a thing that it possesses by its very nature and those that it has merely accidentally. For example, it lies in the nature of water to be composed of hydrogen and oxygen—being composed in this way is part of what it is to be water—but it is merely accidental to water that we use it to brush our teeth. A proposition is metaphysically necessary just in case it is true in virtue of the natures of things. (Also see Kment 2014, chs. 6–7.) Other philosophers (Rayo 2013, §2.2.1, ch. 5; Dorr forthcoming) have discussed the idea that metaphysical necessity can be explained in terms of the idiom "To be F is to be G" (as in "To be water is to be H2O"). Yet another account ties the metaphysical notion of necessity constitutively to causation and explanation (Kment 2006a,b, 2014, 2015a; also see the exchange between Lange 2015 and Kment 2015b).

Dualism requires us to dismantle the picture of inquiry, information and content sketched in the introduction to section 1. Note that it is natural for a dualist to distinguish the space of metaphysically possible worlds from the space of epistemically possible worlds, i.e., from the space of (maximally specific) ways the world might be that cannot be ruled out on the basis of ideal reasoning alone, without empirical evidence (Soames 2005, 2011). The range of epistemically possible worlds outstrips the range of metaphysically possible worlds: there are some ways the world couldn't have been, but which cannot be ruled by ideal reasoning alone. For example, there is no metaphysically possible world where gold has atomic number 78. But prior to carrying out the right chemical investigations, we don't have enough evidence to exclude all scenarios where gold has that atomic number, so some worlds where gold has atomic number 78 are epistemically possible. Empirical evidence is not used only to rule out (metaphysical) possibilities, but is sometimes needed to rule out metaphysical impossibilities that are epistemically possible. Consequently, we cannot in general identify information with sets of metaphysically possible worlds, since we need to distinguish between states of information in which the available evidence rules out the same metaphysically possible worlds but different metaphysically impossible worlds. By the same token, the information encoded in a sentence cannot in general be identified with a set of metaphysically possible worlds, since two sentences may be true in all the same metaphysically possible worlds, but not in all the same epistemically possible worlds. If we wanted to identify information and sentential contents with sets of worlds, it would seem more promising to use sets of epistemically possible worlds. But the dualist may instead reject the possible-worlds account of information and propositions altogether (see, e.g., Soames 1987, 2003, 395f.).

9.2.3 Monism

As mentioned above, monists explain the data described by Kripke by holding that the sentences that figure in Kripke's examples are associated with two different propositions, one that is necessary and another that is contingent. This view comes in two main versions. According to the first version, both propositions are semantically expressed by the sentence. Proponents of this account need to formulate a semantic theory that explains how that is possible. According to the second version, only one of these propositions is semantically expressed by the sentence, while the other is the proposition that is communicated by a typical assertoric use of the sentence. A philosopher holding this view needs to explain the pragmatic mechanism by which an utterance of the sentence comes to communicate the second proposition.

The first version of monism has been developed by David Chalmers and Frank Jackson (Chalmers 1996, 1999, 2002a,b, 2004, 2006a,b; Chalmers and Jackson 2001; Jackson 1998, 2004, 2011), who build on earlier work by David Kaplan (1989a,b), Gareth Evans (1979) and Martin Davies and Lloyd Humberstone (1980), and others. On Chalmers's and Jackson's view, what explains the phenomena uncovered by Kripke is not a difference between two spaces of possible worlds. There is only a single space of possible worlds: the metaphysically possible worlds—the ways the world could have been—just are the epistemically possible worlds: the ways the world might be for all we can know independently of empirical evidence. What explains the data is a difference between two different ways in which sentences can be used to describe the worlds in that space, i.e., between two different notions of a sentence's being true in a world. The distinction can be illustrated by appeal to our example of the proper name 'Phosphorus.' Suppose that we have just introduced this name by using the description 'the last celestial body visible in the morning' to fix its reference. Consider a possible world ww where the description singles out, not Venus (as in our world), but Saturn. Assume further that in ww (as in the actual world), Venus is the second planet from the sun, but Saturn is not. Consider:

- (5)Phosphorus is the second planet from the sun.

Is (5) true in ww? There are two different ways of understanding this question. On the one hand, it could mean something roughly like this: if ww actually obtains (contrary to what astronomers tell us), is Phosphorus the second planet from the sun? The answer to that question is surely 'no.' 'Phosphorus' refers to whatever is actually the last celestial body visible in the morning, and on the assumption that ww actually obtains, that object is Saturn, and is therefore not the second planet. As Chalmers would put it, (5) is not true at w considered as actual.[5] But we can also interpret the question differently: if ww had obtained, then would Phosphorus have been the second planet? In considering that question, we are not hypothetically assuming that the object that actually satisfies the reference-fixing description is Saturn. Instead, we can draw freely on our belief that the object actually fitting the description is Venus, so that the name picks out Venus in all possible worlds. Since Venus is the second planet in ww, it is true to say: if ww had obtained, then Phosphorus would have been the second planet. In Chalmers's terminology, (5) is true at w considered as counterfactual.

The distinction between the two concepts of truth in a world can be explained within a theoretical framework known as two-dimensional semantics, which assigns to a sentence like (5) an intension that is a function, not from worlds to truth-values, but from pairs of worlds to truth-values. The intension of (5) is the function that assigns the true to a pair of worlds ⟨u;w⟩⟨u;w⟩ just in case the object that is the last celestial body visible in the morning in uu is the second planet in ww.[6] This account makes it easy to define the two notions of truth in a world. A sentence PP is true in ww considered as actual just in case the two-dimensional function assigns the true to ⟨w;w⟩⟨w;w⟩. PP is true in ww considered as counterfactual just in case, where uu is the actual world, the two-dimensional function assigns the true to ⟨u;w⟩⟨u;w⟩. Note that the two-dimensional intension of (5) determines whether (5) is true at a world ww considered as actual. But it does not in general determine whether (5) is true at ww considered as counterfactual. That also depends on which world is actual. Knowledge of a sentence's two-dimensional intension is therefore not in general sufficient to know whether the sentence is true at ww considered as counterfactual. Further empirical evidence may be required.

When combined with the conception of a sentence's content as the set of worlds where it is true, the distinction between the two concepts of truth in a world yields a distinction between two different propositions expressed by a sentence. The first of these propositions is the function that assigns the true to a world ww just in case the sentence is true in ww considered as actual, while the second proposition is the function that assigns the true to a world ww just in case the sentence is true in ww considered as counterfactual. Jackson calls the former proposition the sentence's 'A-intension' (for 'actual') and the latter its 'C-intension' (for 'counterfactual'), while Chalmers calls the former the 'primary intension' and the latter the 'secondary intension.' The distinction between the two propositions expressed by a sentence yields a distinction between two notions of sentential necessity: primary necessity, which applies to sentences with necessary primary intensions, and secondary necessity, which applies to sentences with necessary secondary intensions. If a sentence has primary necessity, then that fact, and a fortiori the fact that the sentence is true, can be read off its two-dimensional intension. Therefore, if we know the two-dimensional intension, then that is enough to know that the sentence is true. No further empirical evidence is required. That motivates the thought that the notion of primary necessity captures the idea of a prioricity or epistemic necessity. The notion of secondary necessity, on the other hand, may be taken to capture the Kripkean idea of metaphysical necessity.

This account makes it straightforward to explain cases of a posteriori necessity: they are simply cases of sentences whose secondary intensions are necessary, but whose primary intensions are contingent. Suppose that 'Hesperus' and 'Phosphorus' were introduced, respectively, by the reference-fixing descriptions 'the first celestial body visible in the evening (if it exists)' and 'the last celestial body visible in the morning (if it exists).' Since the two descriptions single out the same object in the actual world, the sentence 'If Hesperus exists, then Hesperus is Phosphorus' is true in all worlds considered as counterfactual, and therefore has a necessary secondary intension. However, in some non-actual worlds, the two descriptions single out different objects. The sentence is false in such a world considered as actual. The primary intension of the sentence is therefore contingent.

An analogous account can be given of Kripke's examples of the contingent a priori: these concern sentences whose primary intensions are necessary and whose secondary intensions are contingent. Assume again that the reference of 'Julius' is fixed by the description 'the inventor of the zip (if such a person exists).' Then in every world considered as actual, the name singles out the person who is the inventor of the zip in that world (if there is such a person) or nothing (if no such person exists in the world). The primary intension of (4) is necessary. However, when we evaluate (4) in a world ww considered as counterfactual, 'Julius' picks out the individual who is the actual inventor of the zip (provided that there actually is such an individual and that he or she exists in w)w). And since there are possible worlds where that individual exists but is not the inventor of the zip, the secondary intension of (4) is contingent.

Chalmers (2002a, 2010) and Jackson (1998) have tried to support their modal monism by arguing that it is gratuitous to postulate two forms of modality, given that all the phenomena pointed out by Kripke can be accommodated by appeal to a single kind of modality. Dualists may reply that the greater simplicity in the view of modality has been achieved only by adding complexity to the semantic theory. That response could be answered by arguing that two-dimensional semantics can be motivated by independent considerations. That, of course, is controversial, as is the general viability of two-dimensional semantics (see the entry Two-Dimensional Semantics for detailed discussion).

In addition, it is not obvious that the view of Chalmers and Jackson can satisfactorily explain all the phenomena discussed in section 1.1. Some commentators have denied that it can give a viable general account of Kripkean examples (see, e.g., Soames 2005; Vaidya 2008; Roca-Royes 2011). In any case, it is clear that the view can only explain how necessity and epistemic necessity can come apart for sentences whose primary and secondary intensions differ. That may be true of the cases considered by Kripke, but it seems doubtful for the other examples considered in section 1.1 (mathematical and metaphysical truths, laws, and principles connecting the physical to the mental). In response, Chalmers has argued that none of the latter cases are genuine examples of the necessary a posteriori (1999, 2002a).

The second version of monism allows us to accommodate the phenomena considered in section 1.1 while staying much closer to the picture sketched in the introduction to section 1. On this view, the data can be explained by appeal to a single space of possible worlds and a single notion of truth in a world. The proposition semantically expressed by a sentence containing a proper name or natural-kind term is a function from individual worlds to truth-values. The proposition expressed by ‘Phosphorus exists,’ e.g., is a function that assigns the true to those worlds where Venus exists and the false to the other worlds. (If the reference of ‘Phosphorus’ was determined by a reference-fixing description together with the facts about which entity meets the description, then that fact itself is not a semantic fact, but a metasemantic one, i.e., it does not concern the question of what the meaning of the word is, but the question of how the meaning of the word is determined.) What explains the impression that a sentence like (1) expresses an a posteriori claim is the fact that the proposition asserted by a typical utterance of the sentence is not the one that is semantically expressed by it, but a different proposition that is contingent and can only be known empirically?

Robert Stalnaker (1978, 2001) has given a detailed account of the pragmatic mechanism by which a contingent proposition comes to be asserted by the utterance of a sentence that semantically expresses a necessary proposition. On his account, linguistic communication evolves in a context characterized by background assumptions that are shared between the participants. These assumptions can be represented by the set of worlds at which they are jointly true, which Stalnaker calls the 'context set.' The point of assertion is to add the proposition asserted to the set of background assumptions and thereby eliminate worlds where it is not true from the context set. To achieve this, every assertion needs to conform to the rule that the proposition asserted is false in some of the worlds that were in the context set before the utterance (otherwise there are no worlds to eliminate) and true in others (since the audience cannot eliminate all worlds from the context set). Now consider a context where the shared background assumptions include the proposition that the references of 'AA' and 'BB' were fixed by certain descriptions but leave open whether the two descriptions single out the same object. Suppose that someone says 'AA is BB.' In every world in the context set, the sentence semantically expresses either a necessary truth (if the two descriptions single out the same object in the world) or a necessary falsehood (if they don't). If the proposition that the speaker intends to assert were the one that is semantically expressed by the sentence, the aforementioned rule would be violated.[7] To avoid attributing this rule violation to the speaker, the audience will construe the utterance as expressing a different proposition, and the most natural candidate is the proposition that the sentence uttered semantically expresses a true proposition. (Stalnaker calls this the 'diagonal proposition.') By exploiting this mechanism of reinterpretation, a speaker can use the sentence to express the diagonal proposition. This proposition is true in just those worlds in the context set where the two descriptions single out the same object. It is clearly a contingent proposition, and empirical evidence is required to know it. Stalnaker suggests an analogous explanation of Kripke's proposed cases of contingent a priori truth (1978, 83f.).

Stalnaker's account of the necessary a posteriori requires that the proposition semantically expressed by the sentence and the proposition that the sentence semantically expresses a truth hold in different worlds in the context set. And that seems to require that the assumptions shared between the participants of the conversation don't determine what proposition is semantically expressed by the sentence. It has been argued that that assumption is implausible in some cases of Kripkean a posteriori necessities (Soames 2005, 96–105). Suppose that I point to the desk in my office in broad daylight and say 'That desk (if it exists) is made of wood.' Unless the context is highly unusual, the shared assumptions, so the argument goes, uniquely determine what proposition is expressed by the sentence.

9.3 METAPHYSICAL AND NOMIC MODALITY

It often seems very natural to use modal terminology when talking about the laws of nature. We are inclined to say that nothing can move faster than light to express the fact that the laws rule out superluminal motion, and to state Newton's First Law by saying that an object cannot depart from uniform rectilinear motion unless acted on by an external force. This motivates the thought that there is a form of necessity associated with the natural laws.[8] It is controversial, however, whether that form of necessity is simply metaphysical necessity, or another kind of necessity. The former view is taken by necessitarians (Swoyer 1982; Shoemaker 1980, 1998; Tweedale 1984; Fales 1993; Ellis 2001; Bird 2005), who believe that the laws (or the laws conditionalized on the existence of the properties mentioned in them) are metaphysically necessary. Contingentists deny that, but many contingentists hold that there is a kind of necessity distinct from metaphysical necessity that is characteristic of the laws (e.g., Fine 2002), and which may be called natural or nomic necessity. It is often assumed that nomic necessity is a weaker form of necessity than metaphysical necessity: it attaches to the laws and to all truths that are metaphysically necessitated by them, so that anything that is metaphysically necessary is also nomically necessary, but not vice versa.

Necessitarians have given several arguments for their position. Here are two.

The argument from causal essentialism (e.g., Shoemaker 1980, 1998). Some philosophers believe that the causal powers that a property confers on its instances are essential to it. Assuming that causal laws describe the causal powers associated with properties, it follows that these laws (or versions of them that are conditionalized on the existence of the relevant properties) are necessary truths. This is, in the first instance, only an argument for the necessity of causal laws, but perhaps it can be argued that all laws of nature are of this kind. Of course, even if this assumption is granted, the argument is only as strong as the premise that properties have their associated causal powers essentially. To support this view, Sydney Shoemaker (1980) has given a battery of epistemological arguments. He points out that our knowledge of the properties that an object possesses can only rest on their effects on us, and must therefore be grounded in the causal powers associated with these properties. But, he goes on to argue that, without a necessary connection between the properties and the associated causal powers, an object's effects on us could not serve as a source of all the knowledge about an object's properties that we take ourselves to possess.

The argument from counterfactual robustness (Swoyer 1982; Fales 1990, 1993; also see Lange 2004 for discussion). Natural laws are often believed to differ from accidental generalizations by their counterfactual robustness (counterfactual-supporting power). If it is a law that all Fs are G, then this generalization would still have been true if there had been more Fs than there actually are, or if some Fs had found themselves in conditions different from the ones that actually obtain. For example, it would still have been true that nothing moves faster than light if there had been more objects than there actually are, or if some bodies had been moving in a different direction. Contrast this with No emerald has ever decorated a royal crown. That may be true, but it is not very robust. It would have been false if some kings or queens of the past had made different decisions. Some necessitarians have argued that contingentism about the laws cannot provide a plausible explanation of the special counterfactual robustness of the laws. Note that a counterfactual “if it had been the case that P, then it would have been the case that Q” is usually taken to be true if Q is true in those metaphysically possible P-worlds that are closest to actuality. On this view, the special counterfactual robustness of the law All Fs are G amounts, roughly speaking, to this: of all the metaphysically possible worlds that contain some additional Fs, or where some actual Fs are in somewhat different circumstances, the ones where the actual law holds are closer than the rest. If the laws hold in some metaphysically possible worlds but not in others, then the reason why the former are closer than the latter must be that the rules we are using for deciding which worlds count as the closest say so. But which such rules we use is a matter of convention. The counterfactual-supporting power of the laws does not seem to be a purely conventional matter, however. Necessitarianism, the argument continues, offers a better explanation: the laws hold in the closest possible worlds simply because they hold in all metaphysically possible worlds. Conventions don’t come into it. The contingentist may reply that, even though the counterfactual robustness of the laws is grounded in a convention, that convention may not be arbitrary, but may have its rationale in certain features of the laws that make them, in some sense, objectively important (Sidelle 2002), e.g., the fact that they relate to particularly pervasive and conspicuous patterns in the history of the world.

Contingentism has often been defended by pointing out that the laws of nature can be known only a posteriori, and that their negations are conceivable (see Sidelle 2002). Necessitarians may reply to the first point that Kripke’s work has given us reasons for thinking that a posteriori truths can be metaphysically necessary (see section 1.1). In response to the second point, they may grant that the negation of a law is conceivable, but deny that conceivability is a good guide to possibility (see the entry Epistemology of Modality). Alternatively, they may deny that we can really conceive of a situation in which, say, bodies violate the law of gravitation. What we can conceive of is a situation in which objects move in ways that appear to violate the law. But that situation cannot be correctly be described as involving objects with mass. Rather, the objects in the imagined situation have a different property that is very similar to mass (call it 'schmass') but which is governed by slightly different laws. Contingentists may reply that the non-existence of schmass (or the non-existence of objects that move in the way imagined) is itself a law, so that we have, after all, conceived of a situation where one of the actual laws fails (see Fine 2002)

9.4 THE STRUCTURE OF THE MODAL REALM

The concepts of metaphysical, epistemic, and nomic necessity are only a few of the modal notions that figure in our thought and discourse (as should be clear from the long list of uses of modal terms given in the introduction to this entry). We also speak of

(6) Practical necessity

Biological necessity

Medical necessity

Moral necessity

Legal necessity

and of a whole lot more. One would expect that some of these modal concepts can be defined in terms of others. But how can that be done? And is it possible to single out a small number of fundamental notions of necessity in terms of which all the others can be defined?

It may be helpful in approaching these questions to distinguish between two salient ways in which one modal property can be defined in terms of another (Fine 2002, 254f.).

Restriction. To say that property N can be defined from kind of necessity N∗ by restriction is to say that a proposition's having N can be defined as the combination of two things: (i) the proposition's having N∗, and (ii) its meeting certain additional conditions.

Relativization / quantifier restriction. To say that a property N can be defined from a kind of necessity N∗ by relativization to a class of propositions S is to say that a proposition's having N can be defined as its being N∗-necessitated by S. A closely related way in which a modal property can be defined in terms of another is by quantifier restriction. Suppose that P∗ is a kind of possibility that is the dual of N∗ (in the sense that it is P∗-possible that p just in case it's not N∗-necessary that not-p), and that we have at our disposal the notion of a P∗-possible world (a world that could P∗-possibly have been actualized). To say that the property N can be defined from N∗ by quantifier restriction is to say that that a proposition's having N can be defined as its being true in all P∗-possible worlds that meet a certain condition C. (This is only the simplest way of defining a modal property from a kind of necessity by quantifier restriction. Much more sophisticated methods have been proposed. See, e.g., Kratzer 1977, 1991.) Given reasonable assumptions, every definition by relativization corresponds to a definition by quantifier restriction, and vice versa.[9]

Restriction allows us to define narrower modal properties from broader ones. For example, it seems natural to hold that mathematical necessity can be defined from metaphysical necessity by restriction. (Perhaps a proposition's being mathematically necessary can be defined as its being both metaphysically necessary and a mathematical truth (Fine 2002, 255), or as its being metaphysically necessary because it is a mathematical truth.) Relativization and quantifier restriction, by contrast, allow us to define broader modal properties in terms of narrower ones. For example, it may be held that biological necessity can be defined as the property of being metaphysically (or perhaps nomically) necessitated by the basic principles of biology.

A modal property N is called alethic just in case the claim that a proposition has N entails that the proposition is true. Metaphysical, epistemic and nomic necessity are all alethic. By contrast, moral and legal necessity are not. It is both morally and legally necessary (i.e., it is required both by morality and by the law) that no murders are committed, even though murders are in fact being committed. A modal property defined by restriction from an alethic kind of necessity must itself be alethic. By contrast, relativization allows us to define non-alethic modal properties from alethic ones, by relativizing to a class of propositions that contains some falsehoods. Similarly, we can define a non-alethic modal property from an alethic one by restricting the quantifier over possible worlds to some class that does not include the actual world. For example, legal necessity can perhaps be defined from metaphysical necessity by restricting the quantifier to worlds where everybody conforms to the actual laws.

The properties listed in (6) can very naturally be called ‘kinds of necessity,’ and in some contexts they are the properties expressed by necessity operators like ‘must’ and ‘could not have been otherwise.’ But that is not true of every property that can be defined from some kind of necessity by relativization or restriction. For example, we can define a property by relativizing metaphysical necessity to the class of truths stated in a certain book, but it would not be natural at all to call this property a kind of necessity. It is not plausible that there is a special form of necessity that attaches to all and only the propositions necessitated by the truths in the book. Similarly, the property defined by restricting metaphysical necessity to the truths about cheddar cheese cannot naturally be called a kind of necessity. There is no form of necessity that applies to just those necessary propositions that deal with cheddar and to none of the others. It is a good question what distinguishes those properties defined by relativization and restriction that we are willing to count as forms of necessity from the rest. Perhaps the most natural answer is that the distinction is dictated by our interests and concerns, and does not reflect a deep metaphysical difference.

A more pressing question is whether some of the forms of necessity discussed in sections 1 and 2 can be defined in terms of the others by relativization or restriction. Consider epistemic and metaphysical necessity first, and suppose for the sake of the argument that dualism is true and the two properties are indeed different forms of necessity. Can one of them be defined in terms of the other by one of the aforementioned methods? Not if there are both necessary a posteriori and contingent a priori propositions, since relativization and restriction only allow us to define one property in terms of another if the extension of one is a subclass of that of the other. However, the existence of contingent a priori truths is more controversial than that of necessary a posteriori propositions, and someone trying to define epistemic necessity in terms of metaphysical necessity or vice versa may repudiate the contingent a priori and hold that the extension of epistemic necessity is included in that of metaphysical necessity. Then such a philosopher could try (a) to define metaphysical necessity from epistemic necessity by relativization to some suitable class, or (b) to define epistemic necessity from metaphysical necessity by restriction.

Such a definition may get the extension of the definiendum right. But a definition may be intended to do much more than that: it may be meant to tell us what it is for something to fall under the concept to be defined. Suppose that someone tried to define the property of being an equiangular triangle as that of being a triangle whose sides are of equal length. While this is extensionally correct, it does not give us the right account of what it is for something to be an equiangular triangle (what it is for something to have that property has something to do with the sizes of its angles, not with the lengths of its sides). It could be argued that definitions of type (a) and (b) face similar difficulties. For example, a definition of kind (a) entails that a proposition's being metaphysically necessary consists in its being epistemically necessitated by a certain class of propositions. But that would make metaphysical necessity an epistemic property, and dualists typically want to resist that idea. Similarly for definitions of type (b). Whether something is epistemically necessary (in the sense of being a priori) seems to be a purely epistemic matter. A priori propositions may also be metaphysically necessary, but their metaphysical necessity isn't part of what makes them a priori, and therefore shouldn't be mentioned in a definition of a prioricity.

If this argument is correct, then it is impossible to define epistemic modal properties in terms of non-epistemic ones, or vice versa. But what about metaphysical and nomic necessity? Suppose for the sake of the argument that there is such a thing as nomic necessity (a form of necessity associated with the laws of nature) but that contingentism about the natural laws is true, so that nomic necessity is indeed distinct from metaphysical necessity. Can we define one of these properties in terms of the other? The most natural way of doing this would be to say that

(7) Nomic necessity can be defined as the property of being metaphysically necessitated by the laws of nature.

Such a definition may be extensionally accurate, and many philosophers would not hesitate to endorse it. But others have doubted that it captures what it is for a proposition to be nomically necessary (Fine 2002). Nomic necessity is a special modal status enjoyed by all and only the propositions that are metaphysically necessitated by the natural laws. Now, if P is metaphysically necessitated by the laws without itself being a law, then it may seem plausible to say, in some sense, that P has that special modal status because P is metaphysically necessitated by the laws. But the reason why being metaphysically necessitated by the laws confers that special modal status on P is presumably that the laws themselves have that modal status and that this modal status gets transmitted across metaphysical necessitation. But if we now ask what makes it so that the laws themselves have that special modal status, (7) does not seem to give us the correct answer: the special necessity of the laws doesn't consist in the fact that they are metaphysically necessitated by the laws. Hence, (7) cannot be a correct general account of what constitutes that special modal status.

It is open to debate which kinds of necessity are fundamental, in the sense that all others can be defined in terms of them, while they are not themselves definable in terms of others. The monist view considered in section 1.3, when combined with (7), may inspire the hope that we can make do with a single fundamental kind of necessity. Others have argued that there are several kinds of necessity that are not mutually reducible. For example, Fine (2002) suggests (in a discussion that sets aside epistemic modality) that there are three fundamental kinds of necessity, which he calls 'metaphysical,' 'nomic' and 'normative' necessity.

The reduction of the various kinds of necessity to a small number of fundamental ones is an important step towards the goal of a unified account of modality. But those who believe that there are several different fundamental kinds of necessity need to address another question: What is the common feature of these fundamental kinds of necessity that makes them all kinds of necessity? Why do they count as kinds of necessity, while other properties don't?

One strategy for answering this question, which centers on non-epistemic forms of necessity, starts from a certain conception of what (non-epistemic) necessity consists in: for a proposition to be necessary is for its truth to be, in a certain sense, particularly firm, secure, inexorable or unshakable in a wholly objective way. A necessary truth could not easily have been false (it could less easily have been false than a contingent truth). We may call this feature of a proposition 'modal force.' It is natural to apply this conception to metaphysical and nomic necessity. Each of these properties may be held to consist in having a certain grade of modal force, though if contingentism is true, the degree of modal force required for nomic necessity is lower than that required for metaphysical necessity. We could then say that a property is one of the fundamental forms of necessity just in case a proposition P's possessing that property consists entirely in P's having a specific grade of modal force. Other kinds of necessity, like those listed in (6) can be defined from the fundamental ones by relativization or restriction. Having these properties does not consist simply in having a specific grade of modal force (and these properties therefore aren't among the fundamental kinds of necessity). For example, if a property is defined by relativizing metaphysical necessity to a class of propositions S, then the fact that a proposition P has that property consists in the fact that the connection between S and P has a certain grade of modal force. But that is not the same thing as P itself having a certain grade of modal force. Similarly, if a property is defined from, say, metaphysical necessity by restriction, then having that property does not consist merely in possessing such-and-such a grade of modal force, but in the conjunction of that feature with some other property.

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. What do you know about the Epistemic and Metaphysical Modality?

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2. Discuss about Metaphysical and Nomic Modality.

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3. What do you understand the Structure of the Modal Realm?

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9.5 LET US SUM UP

This approach evidently leaves the question how to understand the idea of modal force (of a proposition's truth being very unshakable). Some authors have attempted to explain this notion in counterfactual terms (see Lewis 1973a, §2.5; Lewis 1973b, §2.1; McFetridge 1990, 150ff.; Lange 1999, 2004, 2005; Williamson 2005, 2008; Hill 2006; Kment 2006a; cp. Jackson 1998, Chalmers 2002a): the necessary truths are distinguished from the contingent ones by the fact that they are not only true as things actually are, but that they would still have been true if things had been different in various ways. To capture this idea more precisely, Lange (2005) introduces the concept of 'stability': a deductively closed set SS of truths is stable just in case, for any claim PP in SS and any claim QQ consistent with SS, it is true in any context to say that it would still have been the case that PP if it had been the case that QQ. The different forms of necessity have in common that their extensions are stable sets.

Kment (2006a, 2014, chs. 1–2) argues that modal force, and hence necessity and possibility, come in many degrees (cp. Williamson 2016). We often talk about such degrees of possibility when we say things like 'Team AA could more easily have won than Team BB,' 'Team AA could easily have won' or 'Team AA almost won.' The first utterance states that AA's winning had a greater degree of possibility than BB's winning, while the second and third simply ascribe a high degree of possibility to AA's winning. A proposition's degree of possibility is the higher the less of a departure from actuality is required for it to be true. Suppose, e.g., that Team AA would have won if one of their players had stood just an inch further to the left at a crucial moment during the game. Then we can truly say that the team could easily have won. More formally, PP's degree of possibility is the higher the closer the closest PP-worlds are to actuality (also see Lewis 1973a, §2.5; Lewis 1973b, §2.1; Kratzer 1991). Similarly, a truth's degree of necessity is measured by the distance from actuality to the closest worlds where it is false. What metaphysical necessity, nomic necessity and the other grades of necessity have in common is that each of them is the property of having a degree of possibility that is above a certain threshold. What distinguishes them is a difference in their associated thresholds.

9.6 KEY WORDS

Variety: the quality or state of being different or diverse; the absence of uniformity or monotony.

Modal Logic: Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. A modal—a word that expresses a modality—qualifies a statement.

9.7 QUESTIONS FOR REVIEW

1. Discuss about The Data concept in Modal Logic.
2. Discuss about Dualism.
3. What is Monism?

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9.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. See Section 9.2
2. See Section 9.3
3. See Section 9.4

UNIT 10: MODAL PROPOSITION

STRUCTURE

10.0 OBJECTIVES

As we know inference is the main subject matter of logic. The term refers to the argument in which a proposition is arrived at and affirmed or denied on the basis of one or more other propositions accepted as the starting point of the process. To determine whether or not an inference is correct the logician examines the propositions that are the initial and end points of that argument and the relationships between them. This clearly denotes the significance of propositions in the study of logic. In this unit you are expected to study:

• the nature • the definition

• the types and forms of propositions

• the difference between propositions and sentences and judgments

• the description of various types of propositions viewed from different standpoints like, composition, generality, relation, quantity, quality, and modality.

After this unit 10, we can able to understand:

- History of Logic and Proposition
- Propositions and Sentences
- Propositions and Judgments

- Types of Proposition
- Quality and Quantity

10.1 INTRODUCTION

Classical logic concerns itself with forms and classifications of propositions. We shall begin with the standard definition of proposition. A proposition is a declarative sentence which is either true or false but not both. Also a proposition cannot be neither true nor false. A proposition is always expressed with the help of a sentence. For example - the same proposition "It is raining" can be expressed in English, Hindi, and Sanskrit and so on. It means that two or more than two sentences may express the same proposition. This is possible only when proposition is taken as the meaning of the sentence which expresses it. Therefore sentence is only the vehicle of or the means of expressing a proposition. It is the unit of thought and logic whereas sentence is the unit of grammar. A sentence may be correct or incorrect; the grammatical rules determine this. A proposition may be true or false, the empirical facts determine the status. The primary thing about a sentence is its grammatical form, but the primary thing about a proposition is its meaning and implication. The different types of sentences are not different types of propositions. Some types of sentences are not propositions at all. Sentences may be assertive, interrogative, and imperative. Only assertive types of sentences are propositions and rest of them are not (for more details, see below 10.3). A set of proposition make up an argument. Let us see what role propositions play and how logicians will be concerned in logic by taking a simple example of an argument: All men are mortal.

proposition1 All kings are men.
proposition2 Therefore all kings are mortal.
proposition3 Given these propositions as true or false, the logician will only find out whether the argument is valid or not by using certain rules that we shall learn later. Before we proceed further, it is of importance that we situate the discussion on "Proposition" in the whole context of the history of Logic itself.

Propositional Logic is concerned with propositions and their interrelationships. The notion of a proposition here cannot be defined precisely. Roughly speaking, a proposition is a possible condition of the world that is either true or false, e.g. the possibility that it is raining, the possibility that it is cloudy, and so forth. The condition need not be true in order for it to be a proposition. In fact, we might want to say that it is false or that it is true if some other proposition is true.

In this chapter, we first look at the syntactic rules that define the language of Propositional Logic. We then introduce the notion of a truth assignment and use it to define the meaning of Propositional Logic sentences. After that, we present a mechanical method for evaluating sentences for a given truth assignment, and we present a mechanical method for finding truth assignments that satisfy sentences. We conclude with some examples of Propositional Logic in formalizing Natural Language and Digital Circuits.

2.2 Syntax

In Propositional Logic, there are two types of sentences -- simple sentences and compound sentences. Simple sentences express simple facts about the world. Compound sentences express logical relationships between the simpler sentences of which they are composed.

Simple sentences in Propositional Logic are often called proposition constants or, sometimes, logical constants. In what follows, we write proposition constants as strings of letters, digits, and underscores ("_"), where the first character is a lower case letter. For example, raining is a proposition constant, as are rAiNiNg, r32aining, and raining_or_snowing. Raining is not a proposition constant because it begins with an upper case character. 324567 fails because it begins with a number. raining-or-snowing fails because it contains hyphens (instead of underscores).

Compound sentences are formed from simpler sentences and express relationships among the constituent sentences. There are five types of compound sentences, viz. negations, conjunctions, disjunctions, implications, and biconditionals.

A negation consists of the negation operator ¬ and an arbitrary sentence, called the target. For example, given the sentence p, we can form the negation of p as shown below.

$(\neg p)$

A conjunction is a sequence of sentences separated by occurrences of the ∧ operator and enclosed in parentheses, as shown below. The constituent sentences are called conjuncts. For example, we can form the conjunction of p and q as follows.

$(p \wedge q)$

A disjunction is a sequence of sentences separated by occurrences of the ∨ operator and enclosed in parentheses. The constituent sentences are called disjuncts. For example, we can form the disjunction of p and q as follows.

$(p \vee q)$

An implication consists of a pair of sentences separated by the ⇒ operator and enclosed in parentheses. The sentence to the left of the operator is called the antecedent, and the sentence to the right is called the consequent. The implication of p and q is shown below.

$(p \Rightarrow q)$

A biconditional is a combination of an implication and a reverse implication. For example, we can express the biconditional of p and q as shown below.

$(p \Leftrightarrow q)$

Note that the constituent sentences within any compound sentence can be either simple sentences or compound sentences or a mixture of the two. For example, the following is a legal compound sentence.

$((p \vee q) \Rightarrow r)$

One disadvantage of our notation, as written, is that the parentheses tend to build up and need to be matched correctly. It would be nice if we could dispense with parentheses, e.g. simplifying the preceding sentence to the one shown below.

p ∨ q ⇒ r

Unfortunately, we cannot do without parentheses entirely, since then we would be unable to render certain sentences unambiguously. For example, the sentence shown above could have resulted from dropping parentheses from either of the following sentences.

((p ∨ q) ⇒ r)

(p ∨ (q ⇒ r))

The solution to this problem is the use of operator precedence. The following table gives a hierarchy of precedences for our operators. The ¬ operator has higher precedence than ∧; ∧ has higher precedence than ∨; ∨ has higher precedence than ⇒; and ⇒ has higher precedence than ⇔.

¬

∧

∨

⇒

⇔

In sentences without parentheses, it is often the case that an expression is flanked by operators, one on either side. In interpreting such sentences, the question is whether the expression associates with the operator on its left or the one on its right. We can use precedence to make this determination. In particular, we agree that an operand in such a situation always associates with the operator of higher precedence. The following examples show how these rules work in various cases. The expressions on the right are the fully parenthesized versions of the expressions on the left.

¬ p ∧ q	((¬ p) ∧ q)
p ∧ ¬q	(p ∧ (¬ q))
p ∧ q ∨ r	((p ∧ q) ∨ r)

$p \vee q \wedge r$	$(p \vee (q \wedge r))$
$p \Rightarrow q \Leftrightarrow r$	$((p \Rightarrow q) \Leftrightarrow r))$
$p \Leftrightarrow q \Rightarrow r$	$(p \Leftrightarrow (q \Rightarrow r))$

When an operand is surrounded by two $\wedge$ operators or by two $\vee$ operators, the operand associates to the left. When an operand is surrounded by two $\Rightarrow$ operators or by two $\Leftrightarrow$ operators, the operand associates to the right.

$p \wedge q \wedge r$	$((p \wedge q) \wedge r))$
$p \vee q \vee r$	$((p \vee q) \vee r))$
$p \Rightarrow q \Rightarrow r$	$(p \Rightarrow (q \Rightarrow r))$
$p \Leftrightarrow q \Leftrightarrow r$	$(p \Leftrightarrow (q \Leftrightarrow r))$

Note that just because precedence allows us to delete parentheses in some cases does not mean that we can dispense with parentheses entirely. Consider the example shown earlier. Precedence eliminates the ambiguity by dictating that the sentence without parentheses is an implication with a disjunction as antecedent. However, this makes for a problem for those cases when we want to express a disjunction with an implication as a disjunct. In such cases, we must retain at least one pair of parentheses.

We end the section with two simple definitions that are useful in discussing Propositional Logic. A propositional vocabulary is a set of proposition constants. A propositional language is the set of all propositional sentences that can be formed from a propositional vocabulary.

2.3 Semantics

The treatment of semantics in Logic is similar to its treatment in Algebra. Algebra is unconcerned with the real-world significance of variables. What is interesting are the relationships among the values of the variables expressed in the equations we write. Algebraic methods are designed to respect these relationships, independent of what the variables represent.

In a similar way, Logic is unconcerned with the real world significance of proposition constants. What is interesting is the relationship among the truth values of simple sentences and the truth values of compound sentences within which the simple sentences are contained. As with Algebra, logical reasoning methods are independent of the significance of proposition constants; all that matter is the form of sentences.

Although the values assigned to proposition constants are not crucial in the sense just described, in talking about Logic, it is sometimes useful to make truth assignments explicit and to consider various assignments or all assignments and so forth. Such an assignment is called a truth assignment.

Formally, a truth assignment for a propositional vocabulary is a function assigning a truth value to each of the proposition constants of the vocabulary. In what follows, we use the digit 1 as a synonym for true and 0 as a synonym for false; and we refer to the value of a constant or expression under a truth assignment i by superscripting the constant or expression with i as the superscript.

The assignment shown below is an example for the case of a propositional vocabulary with just three proposition constants, viz. p, q, and r.

pi = 1

qi = 0

ri = 1

The following assignment is another truth assignment for the same vocabulary.

pi = 0

qi = 0

ri = 1

Note that the formulas above are not themselves sentences in Propositional Logic. Propositional Logic does not allow superscripts and does not use the = symbol. Rather, these are informal, metalevel statements about particular truth assignments. Although talking about Propositional Logic using a notation similar to that Propositional Logic can sometimes be confusing, it allows us to convey meta-information precisely and efficiently. To minimize problems, in this book we use such meta-notation infrequently and only when there is little chance of confusion.

Looking at the preceding truth assignments, it is important to bear in mind that, as far as logic is concerned, any truth assignment is as good as any other. Logic itself does not fix the truth assignment of individual proposition constants.

On the other hand, given a truth assignment for the proposition constants of a language, logic does fix the truth assignment for all compound sentences in that language. In fact, it is possible to determine the truth value of a compound sentence by repeatedly applying the following rules.

If the truth value of a sentence is true, the truth value of its negation is false. If the truth value of a sentence is false, the truth value of its negation is true.

2.4 Evaluation

Evaluation is the process of determining the truth values of compound sentences given a truth assignment for the truth values of proposition constants.

As it turns out, there is a simple technique for evaluating complex sentences. We substitute true and false values for the proposition constants in our sentence, forming an expression with 1s and 0s and logical operators. We use our operator semantics to evaluate subexpressions with these truth values as arguments. We then repeat, working from the inside out, until we have a truth value for the sentence as a whole.

As an example, consider the truth assignment i shown below.

$p^i = 1$

$q^i = 0$

$r^i = 1$

Using our evaluation method, we can see that i satisfies $(p \vee q) \wedge (\neg q \vee r)$.

$(p \lor q) \land (\neg q \lor r)$

$(1 \lor 0) \land (\neg 0 \lor 1)$

$1 \land (\neg 0 \lor 1)$

$1 \land (1 \lor 1)$

$1 \land 1$

1

Now consider truth assignment j defined as follows.

$pj = 0$

$qj = 1$

$rj = 0$

In this case, j does not satisfy $(p \lor q) \land (\neg q \lor r)$.

$(p \lor q) \land (\neg q \lor r)$

$(0 \lor 1) \land (\neg 1 \lor 0)$

$1 \land (\neg 1 \lor 0)$

$1 \land (0 \lor 0)$

$1 \land 0$

0

Using this technique, we can evaluate the truth of arbitrary sentences in our language. The cost is proportional to the size of the sentence. Of course, in some cases, it is possible to economize and do even better. For example, when evaluating a conjunction, if we discover that the first conjunct is false, then there is no need to evaluate the second conjunct since the sentence as a whole must be false.

2.5 Satisfaction

Satisfaction is the opposite of evaluation. We begin with one or more compound sentences and try to figure out which truth assignments satisfy those sentences. One nice feature of Propositional Logic is that there are effective procedures for finding truth assignments that satisfy Propositional Logic sentences. In this section, we look at a method based on truth tables.

A truth table for a propositional language is a table showing all of the possible truth assignments for the proposition constants in the language.

The columns of the table correspond to the proposition constants of the language, and the rows correspond to different truth assignments for those constants.

The following figure shows a truth table for a propositional language with just three proposition constants (p, q, and r). Each column corresponds to one proposition constant, and each row corresponds to a single truth assignment. The truth assignments i and j defined in the preceding section correspond to the third and sixth rows of this table, respectively.

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Note that, for a propositional language with n proposition constants, there are n columns in the truth table and 2n rows.

In solving satisfaction problems, we start with a truth table for the proposition constants of our language. We then process our sentences in turn, for each sentence placing an x next to rows in the truth table corresponding to truth assignments that do not satisfy the sentence. The truth assignments remaining at the end of this process are all possible truth assignments of the input sentences.

As an example, consider the sentence $p \vee q \Rightarrow q \wedge r$. We can find all truth assignments that satisfy this sentence by constructing a truth table for p, q, and r. See below. We place an x next to each row that does not satisfy the sentence (rows 2, 3, 4, 6). Finally, we take the remaining rows (1, 5, 7, 8) as answers.

p	q	r
1	1	1

x	1	1	0	x
x	1	0	1	x
x	1	0	0	x
	0	1	1	
x	0	1	0	x
	0	0	1	
	0	0	0	

The disadvantage of the truth table method is computational complexity. As mentioned above, the size of a truth table for a language grows exponentially with the number of proposition constants in the language. When the number of constants is small, the method works well. When the number is large, the method becomes impractical. Even for moderate sized problems, it can be tedious. Even for an application like Sorority World, where there are only 16 proposition constants, there are 65,536 truth assignments.

Over the years, researchers have proposed ways to improve the performance of truth table checking. However, the best approach to dealing with large vocabularies is to use symbolic manipulation (i.e. logical reasoning and proofs) in place of truth table checking. We discuss these methods in Chapters 4 and 5.

2.6 Example - Natural Language

As an exercise in working with Propositional Logic, let's look at the encoding of various English sentences as formal sentences in Propositional Logic. As we shall see, the structure of English sentences, along with various key words, such as if and no, determine how such sentences should be translated.

The following examples concern three properties of people, and we assign a different proposition constant to each of these properties. We use the constant c to mean that a person is cool. We use the constant f to mean that a person is funny. And we use the constant p to mean that a person is popular.

As our first example, consider the English sentence If a person is cool or funny, then he is popular. Translating this sentence into the language of Propositional Logic is straightforward. The use of the words if and then suggests an implication. The condition (cool or funny) is clearly a disjunction and the conclusion (popular) is just a simple fact. Using the vocabulary from the last paragraph, this leads to the Propositional Logic sentence shown below.

$c \lor f \Rightarrow p$

Next, we have the sentence A person is popular only if he is either cool or funny. This is similar to the previous sentence, but the presence of the phrase only if suggests that the conditionality goes the other way. It is equivalent to the sentence If a person is popular, then he is either cool or funny. And this sentence can be translated directly into Propositional Logic as shown below.

$p \Rightarrow c \lor f$

A person is popular if and only if he is either cool or funny. The use of the phrase if and only if suggests a biconditional, as in the translation shown below. Note that this is the equivalent to the conjunction of the two implications shown above. The biconditional captures this conjunction in a more compact form.

$p \Leftrightarrow c \lor f$

Finally, we have a negative sentence. There is no one who is both cool and funny. The word no here suggests a negation. To make it easier to translate into Propositional Logic, we can first rephrase this as It is not the case that there is a person who is both cool and funny. This leads directly to the following encoding.

$\neg(c \land f)$

Note that, just because we can translate sentences into the language of Propositional Logic does not mean that they are true. The good news is that we can use our evaluation procedure to determine which sentences are true and which are false?

Suppose we were to imagine a person who is cool and funny and popular, i.e. the proposition constants c and f and p are all true. Which of our sentences are true and which are false.

Using the evaluation procedure described earlier, we can see that, for this person, the first sentence is true.

$c \lor f \Rightarrow p$

$(1 \lor 1) \Rightarrow 1$

$1 \Rightarrow 1$

1

The second sentence is also true.

$p \Rightarrow c \lor f$

$1 \Rightarrow (1 \lor 1)$

$1 \Rightarrow 1$

1

Since the third sentence is really just the conjunction of the first two sentences, it is also true, which we can confirm directly as shown below.

$p \Leftrightarrow c \lor f$

$1 \Leftrightarrow (1 \lor 1)$

$1 \Leftrightarrow 1$

1

Unfortunately, the fourth sentence is not true, since the person in this case is both cool and funny.

$\neg(c \land f)$

$\neg(1 \land 1)$

$\neg 1$

0

In this particular case, three of the sentences are true, while one is false. The upshot of this is that there is no such person (assuming that the theory expressed in our sentences is correct). The good news is that there are cases where all four sentences are true, e.g. a person who is cool and popular but not funny or the case of a person who is funny and popular but not cool. Question to consider: What about a person is neither cool nor funny nor popular? Is this possible according to our theory? Which of the sentences would be true and which would be false?

2.7 Example - Digital Circuits

Now let's consider the use of Propositional Logic in modeling a portion of the physical world, in this case, a digital circuit like the ones used in building computers.

The diagram below is a pictorial representation of such a circuit. There are three input nodes, some internal nodes, and two output nodes. There are five gates connecting these nodes to each other - two xor gates (the gates on the top), two and gates (the gates on the lower left), and one or gate (the gate on the lower right).

p

q

o

r

a

b

s

c

Click on p, q, r to toggle their values.

At a given point in time, a node in a circuit can be either on or off. The input nodes are set from outside the circuit. A gate sets its output either on or off based on the type of gate and the values of its input nodes. The output of an and gate is on if and only if both of its inputs are on. The value of an or node is on if and only if at least one of its inputs is on. The output of an xor gate is on if and only if its inputs disagree with each other.

Given the Boolean nature of signals on nodes and the deterministic character of gates, it is quite natural to model digital circuits in Propositional Logic. We can represent each node of a circuit as a proposition constant, with the idea that the node is on if and only if the constant is true. Using the language of Propositional Logic, we can capture the behavior of gates by writing sentences relating the values of the inputs nodes and out nodes of the gates.

The sentences shown below capture the five gates in the circuit shown above. Node o must be on if and only if nodes p and q disagree.

$(p \wedge \neg q) \vee (\neg p \wedge q) \Leftrightarrow o$

$r \wedge o \Leftrightarrow a$

$p \wedge q \Leftrightarrow b$

$(o \wedge \neg r) \vee (\neg o \wedge r) \Leftrightarrow s$

$a \vee b \Leftrightarrow c$

Once we have done this, we can use our formalization to analyze the circuit - to determine if it meets it specification, to test whether a particular instance is operating correctly, and to diagnose the problem in cases here it is not.

Recap

The syntax of Propositional Logic begins with a set of proposition constants. Compound sentences are formed by combining simpler sentences with logical operators. In the version of Propositional Logic used here, there are five types of compound sentences - negations, conjunctions, disjunctions, implications, and biconditionals. A truth assignment for Propositional Logic is a mapping that assigns a truth value to each of the proposition constants in the language. A truth assignment satisfies a sentence if and only if the sentences is true under that truth assignment according to rules defining the logical operators of the language. Evaluation is the process of determining the truth values of a complex sentence, given a truth assignment for the truth values of proposition constants in that sentence. Satisfaction is the process of determining whether or not a sentence has a truth assignment that satisfies it.

10.2 HISTORY OF LOGIC AND PROPOSITION

Aristotle, the classical logician defines proposition as that which contains subject, predicate and a copula. “Rose is red” is a proposition. Here ‘Rose’ is the subject, ‘red’ is the predicate and ‘is’ is the copula. A subject is that about which something is said, a predicate is what is said about the subject and the copula is the link. Further, according to classical logicians copula should be expressed in the form of present tense only. That is why classical logicians talk of reduction of sentences into propositions. According to them all propositions are sentences but all sentences are not propositions. Subject-predicate logic ultimately gave rise to substance-attribute metaphysics in philosophy. Aristotle classifies proposition into four types. They are as follows: Universal affirmative (A); Universal negative (E); Particular affirmative (I) and Particular negative (O). These propositions are called categorical or unconditional propositions because no condition is stated anywhere in the propositions. Letters within parentheses are standard symbols of respective propositions which are extensively used throughout our study of logic. "All men are mortal" is an example of 'A' proposition. "No men are immortal" is an instance of 'E' proposition. "Some men are intelligent" is an 'I' proposition and "Some men are honest" is an instance of 'O' proposition. Aristotle was the first thinker to devise a logical system. He holds that a proposition is a complex involving two terms, a subject and a predicate. The logical form of a proposition is determined by its quantity (universal or particular) and quality (affirmative or negative). The analysis of logical form, types of inference, etc. constitute the subject matter of logic. Aristotle may also be credited with the formulation of several metalogical propositions, most notably the Law of Noncontradiction, the Principle of the Excluded Middle, and the Law of Bivalence. These are important in his discussion of modal logic and tense logic. Aristotle referred to certain principles of propositional logic and to reasoning involving hypothetical propositions. He also formulated nonformal logical theories, techniques and strategies for devising arguments (in the Topics), and a theory of fallacies (in the Sophistical Refutations). Aristotle's pupils Eudemus and Theophrastus modified and developed Aristotelian logic in several ways. The next major innovations in logic are due to the Stoic school. They developed an alternative account of the syllogism, and, in the course of so doing, elaborated a full propositional logic which complements Aristotelian logic. They also investigated various logical antinomies, including the Liar Paradox. The leading logician of this school was Chrysippus, credited with over a hundred works in logic. There were few developments in logic in the succeeding periods, other than a number of handbooks, summaries, translations, and commentaries, usually in a simplified and combined form. The more influential authors include Cicero, Porphyry, and Boethius in the later Roman Empire; the Byzantine scholiast Philoponous; and alFarabi, Avicenna, and Averroes in the Arab world. The next major logician of proposition is Peter Abelard, who worked in the early twelfth century. He composed an independent treatise on logic, the Dialectica, and wrote extensive commentaries. There are discussions of conversion, opposition, quantity, quality, tense logic, a reduction of de dicto to de re modality, and much else. Abelard also clearly formulates several semantic principles. Abelard is responsible for the clear formulation of a pair of relevant criteria for logical consequences. The failure of his criteria led later logicians to reject relevance implication and to endorse material implication. Spurred by Abelard's teachings and problems he proposed, and by further translations, other logicians began to grasp the details of Aristotle's texts. The result, coming to fruition in the middle of the thirteenth century, was the first phase of supposition theory, an elaborate doctrine about the reference of terms in various propositional contexts. Its development is preserved in handbooks by Peter of Spain, Lambert of Auxerre, and William of Sherwood. The theory of obligations, a part of non-formal logic, was also invented at this time. Other topics, such as the relation between time and modality, the conventionality of semantics, and the theory of truth, were investigated. The fourteenth century is the apex of mediæval logical theory, containng an explosion of creative work. Supposition theory is developed extensively in its second phase by logicians such as William of Ockham, Jean Buridan, Gregory of Rimini, and Albert of Saxony. Buridan also elaborates a full theory of consequences, a cross between entailments and inference rules. From explicit semantic principles, Buridan constructs a detailed and extensive investigation of syllogistic, and offers completeness proofs.

10.3 PROPOSITIONS AND SENTENCES

Propositions are stated using sentences. However, all sentences are not propositions. Let's look at a few examples of sentences:

1. Snakes are poisonous. 2. Some students are intelligent. 3. How old are you? 4. May God bless you! 5. What a car! 6. Vote for me. The first two statements are assertions and we can say of these statements that they may either be true or false. Therefore they are propositions. However, we cannot say whether or not the question, 'How old are you?' is true or false. The answer to the question, 'I am 16 years old' may be true or false. The question is not a proposition, while the answer is a proposition. 'May God bless you' is a ceremonial statement and it is neither true nor false. Therefore, such statements are not propositions. 'What a car!' is exclamatory and has nothing to do with being true or false. Exclamatory statements are not propositions. 'Vote for me' is an appeal or command. We cannot attribute truth or falsity to it. Therefore, evocative statements are not propositions. We therefore need to distinguish between sentences and propositions. The differences are: 1. Propositions must be meaningful (meaningful in logical sense) sentences. 2. Propositions must have a subject, a predicate and a word joining the two, a sentence need not. 3. All propositions are either true or false, but sentences may or may not be. 4. Propositions are units of Logic, sentences are units of Grammar.

10.4 PROPOSITIONS AND JUDGMENTS

Till the nineteenth century, idealistic philosophers used the word, 'Judgment' instead of 'propositions'. Nowadays, a distinction is made between the two words. "Judgment" means 'pronouncing a formal decision'. "Proposition" means 'the result of judging'. Judgment is basically the attitude we take whereas proposition is that which we affirm or deny, accept or reject as true or false. Judgment is a mental act, a process, and an event in time. Proposition is time invariant. When we say 'All kings are mortal', it is a proposition. When we assert 'We believe that all kings are mortal', we are in fact taking an attitude, making a judgment. Sometimes, a statement may appear by itself to be a proposition. However, if one knows the context in which the statement is made, it may turn out that the proposition is really a judgment made.

Consider the statement: 'All foreigners are unacceptable'. By itself, it looks like a proposition, but what, if a speech is made and at the end the speaker concludes logically why 'all foreigners are unacceptable'. In such a case the speaker is actually passing a judgment. Sometimes, therefore, we need the context to distinguish a proposition from a judgment. It is only in the beginning of twentieth century that A.N. Whitehead and Bertrand Russell recognize varieties of propositions. According to them subject-predicate logic is only one form of propositions.

10.5 TYPES OF PROPOSITION

Propositions can be viewed from different standpoints and classified into different types:

STANDPOINT	TYPES OF PROPOSITIONS
Composition	Simple, Complex or Compound
Generality	Singular, General
Relation	Categorical, Conditional
Quantity	Universal, Particular
Quality	Affirmative, Negative
Modality	Necessary, Assertoric, Problematic
Significance	Verbal, Real

Composition - Simple Propositions Examples: Love is happiness. Tiger is ferocious. All white men were dreaded by the red Indians. A simple proposition has only one subject and one predicate. Note that the subject 'All white men' is one subject though it has many words. Similarly 'Red Indians' is one predicate.

Composition – Complex or Composite Propositions Examples: Violence does not pay and leads to unhappiness. She is graceful but cannot act. Either he is honest or dishonest. If John comes home, then you must cook chicken. 'She is graceful' is a simple proposition. 'Cannot act' can be written as 'She cannot act', which is a simple proposition again. These simple propositions are connected by a conjunction 'but'. When two or more simple propositions are combined into a single statement we get a complex or composite proposition.

Generality: Singular proposition Examples: The dog wags its tail. George is my friend.

Kapil Dev is a good cricketer. When in a proposition the subject refers to a definite, single object, the proposition is said to be singular proposition. A proper noun or a common noun preceded by a definite article 'the' forms the subject of such a proposition.

Generality - General Propositions Examples: Children like chocolate. All hill stations are health resorts. Some people are funny. Few bikes come with fancy fittings. When in a proposition the subject refers to many objects, the proposition is said to be a general proposition. A common noun forms the subject of such propositions. When it is singular, the indefinite article 'a' is used. 'A dog' means any dog. It generalizes across all dogs. Words like 'some', 'few' refer to more than one object.

Relation - Categorical Propositions Examples: The pillows are soft Junk food is not good for health Music is the food of love.

A proposition that affirms or denies something without any condition is called a categorical proposition. Recall that a proposition has a subject, a predicate and a joining word. The joining word relates the two together. In the first example the subject, "the pillows" is joined to the predicate "soft" by the joining word "are'. In this proposition the softness of the pillow is asserted or affirmed. In the second example it is denied that junk food is good for health. Simple and general propositions are categorical in nature. In the above examples there are no conditions relating the subject and the predicate. Therefore they are called categorical propositions.

Relation: Conditional Propositions Examples: If you study hard, then you will do well. Robert is either an athlete or a carpenter. A conditional proposition consists of two categorical propositions that are so related to each other that one imposes a condition that must be fulfilled if what the other asserts is to be acceptable. There are three types of conditional propositions:

1. Hypothetical proposition

2. Alternative proposition

3. Disjunctive proposition

1. Hypothetical Proposition Examples: If (you are hungry), then (you can eat chocolates.) If (it doesn't rain), then (the harvest will be poor.) A hypothetical proposition consists of two categorical propositions. They are put within parentheses. The first part is called antecedent and the second part is called consequent. These two propositions are related in such a way that if the first is true then the second must be true if the second is false, then the first also is false. However, if the first part is false, the second part may be true or may be false. Example:
If the sun shines then there is light --------------- --------------- antecedent consequent

2. Alternative Proposition Examples: John is either a professor or a musician Either we play football or we play cricket John is either a doctor or the author of this book. An alternative proposition consists of two simple categorical proposition connected by 'either – or' and thus suggesting that any one of these two proposition may be true or both may be true. John may be a professor or may be a musician. It is also likely that John is both a professor and a musician. The two parts of an alterative proposition are known as alternant. Either alternant may be true or both may be true. The alternative proposition will be false only when both the alternant are false.

Either (Alternant)	***Or (Alternant)***	***Proposition***
John is a professor	John is a musician	
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

3. Disjunctive Proposition Examples: It is not the case that both he is honest and he is dishonest. It is not the case that both the meat is boiled and roasted A disjunctive proposition consists of two simple categorical propositions (alternant) which are so related that both cannot be simultaneously true.

Note: The fact that both cannot be true at the same time is the only difference between an alternative and disjunctive proposition. Thus there may be examples which are common to both. In symbolic logic we use disjunctive for alternative and the third variety is called negation. Examples: Either he is in the class or he is in the playground.

Either (Alternant)	***Or (Alternant)***	***Proposition***
	John is in class	John is in playground
FALSE	TRUE	TRUE
TRUE	FALSE	TRUE

Modality: Assertoric Proposition: Examples: The earth moves round the sun. Objects far away appear small to the eyes. At zero degree centigrade water turns into ice. Eleven players form a cricket team. The earth is not perfectly round. When the claim or assertion made in a proposition is verifiable it is called an assertoric proposition. The assertion that the earth moves round the sun can be verified by scientific methods. If the result of such verification is true then the proposition is true.

Modality: Necessary Proposition: Examples: Bachelors are unmarried male. The result of any number multiplied by zero is zero. A point has no dimension. Propositions which are always true by definition are called necessary propositions.

Modality: Problematic Proposition: Examples: Perhaps he is a rich man. She may be happier off with him. There may be famine this year. In a problematic proposition we only guess the truth or falsity and make no definite assertion.

Quantity - Universal Proposition: Examples: All boys in the team are educated. No politicians are honest. Shillong is a hill station.

When the predicate tells something about the entire class referred to by the subject term, it is called a universal proposition. The predicate term 'educated' refers to the entire class referred to by the subject term 'all boys in the team'.

Quantity - Particular Proposition: Examples: Some girls are beautiful. Some songs are classical. Some men are religious. When the predicate term tells something about an indefinite part of the class referred to by the subject term, it is called particular proposition.

Quality: The early discussion on proposition from the standpoint of quantity was based on the subject class being quantified by the word all, some, no etc. When we discuss proposition from the standpoint of quality our focus will be on the 'copula' between the terms. A copula relates the two terms and is of some form of the verb 'to be'- 'is', 'are', 'is not', 'are not' The copula either affirms or denies the relation between two terms

Quality: Affirmative Proposition Examples: Some fruits are sweet. All computers are fast. Mr.John is bald. If the relation between the subject term and the predicate term is positive (or affirmative), the proposition is said to be affirmative. In this case the copula is of the form 'is' or 'are'.

Quality: Negative Proposition: Examples: Some fruits are not sweet. All computers are not fast. Mr. John is not bald. If the relation between the subject term and the predicate term is negative (or denied), the proposition is said to be negative. In this case the copula is of the form 'is not' or 'are not'

10.6 QUALITY AND QUANTITY

So far we have viewed a proposition from various standpoints like composition, relation, modality and so on. More important of these are the standpoints of quality and quantity in viewing categorical propositions. Recall that:

Quantity: Universal
Particular

Quality: Affirmative

Negative

If we view a proposition from a combined stand point of quality and quantity, we get the following classification as in Aristotle's logic: Quality Classification Forms of Proposition 1. Universal+ Affirmative A All (…) are/is (…) 2. Universal+ Negative E No (…) are/is (…) 3. Particular+ Affirmative I Some (…) are (…) 4. Particular+ Negative O Some (..) are not (..)

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1) What is a proposition? Distinguish it from sentence.

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2) Mention Aristotelian classification of proposition.

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10.7 LET US SUM UP

In the above unit we have seen how important it is to reduce sentences to its logical form, namely propositions. However, while changing sentences to propositional forms the following points must be remembered.

1 The meaning of the original sentence must be faithfully preserved in the logical form too.

2 The proposition must express all its three parts in the proper order, viz. subject, copula and predicate.

3 The subject of the proposition can be found out by answering the question “Of what anything is being stated”

4 There must be a copula connecting subject and predicate.

5 When reducing a negative sentence to logical form. The sign of negation should go with the copula and with the predicate of the proposition.

6 Compound sentences must be split up in to simple sentences to construct propositions out of them.

7 The quantity of the propositions must be indicated clearly.

10.8 KEY WORDS

Evocation: Evocation is the act of calling or summoning a spirit, demon, god or other supernatural agent, in the Western mystery tradition. Comparable practices exist in many religions and magical traditions.

Reduction: Reduction in philosophy is the process by which one object, property, concept, theory, etc., is shown to be entirely dispensable in favor of another.

10.9 QUESTIONS FOR REVIEW

1. Discuss the History of Logic and Proposition.
2. Write in details about Propositions and Sentences.
3. What is Propositions and Judgments?
4. What are the Types of Proposition?
5. Discuss about Quality and Quantity.

10.10 SUGGESTED READINGS AND REFERENCES

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- Nath Roy, Bhola. Text Book of Deductive Logic. Culculta: S.C. Sarkar and sons Private Ltd, 1984.

10.11 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1 A proposition is the unit of thought and logic and carriers a definite truth-value. A proposition is expressed with the help of a sentence. While proposition is the unit of thought, sentence is the unit of grammar. The primary thing about the proposition is its logical form while for a sentence its primary thing is its grammatical form.

2 Aristotle has classified proposition into 4 kinds. They are as follows: 1 Universal affirmative (A Proposition) 2 Universal negative (E Proposition) 3 Particular affirmative (I Proposition) 4 Particular negative (O Proposition)

UNIT 11: MODEL PROPOSITIONAL CALCULUS

STRUCTURE

11.0 OBJECTIVES

After this unit, we can able to know:

- To discuss the Pre-History
- To know the Logic of Relatives
- To discuss the Propositional Functions and the Birth of Mathematical Logic
- To discuss about the Fregean Functions and Concepts
- To describe The Emergence of Propositional Functions
- To know about the Propositional Functions in Simple Type Theory
- To discuss the Propositional Functions in Ramified Type Theory
- What is a Propositional Function in Russell?

- To know about the Possible Worlds and Propositional Functions
- To discuss the Montague Semantics
- To discuss the Categorial Grammar

11.1 INTRODUCTION

As the name suggests, propositional functions are functions that have propositions as their values. Propositional functions have played an important role in modern logic, from their beginnings in Frege's theory of concepts and their analyses in Russell's works, to their appearance in very general guise in contemporary type theory and categorial grammar.

In this article, I give an historical overview of the use of propositional functions in logical theory and of views about their nature and ontological status.

11.2 PRE-HISTORY

Before we begin our discussion of propositional functions, it will be helpful to note what came before their introduction. In traditional logic, the role of propositional functions is approximately held by terms. In traditional logic, statements such as ‘dogs are mammals’ are treated as postulating a relation between the terms ‘dogs’ and ‘mammals’.

A term is treated either extensionally as a class of objects or intensionally as a set of properties. The ‘intent’ of the term ‘dog’ includes all the properties that are included in the intent of ‘mammal’. The intensional treatment of ‘dogs are mammals’ interprets this sentence as true because the semantic interpretation of the subject is a superset of the interpretation of the predicate. On the extensional treatment of the sentence, however, the sentence is true because the interpretation of the subject (the class of dogs) is a subset of the interpretation of the predicate (the set of mammals).

These two treatments of the predicate are typical of the two traditions in traditional logic—the intensional and the extensional traditions. Logicians who can be counted among the intensional logicians are Gottfried Leibniz, Johann Lambert, William Hamilton, Stanley Jevons, and Hugh MacColl. Among the extensional logicians are George Boole, Augustus De Morgan, Charles Peirce, and John Venn.

The treatment of terms in the intensional logic tradition property of certain sentences might seem strange to modern readers. The intension of a predicate, in 20th Century philosophy, includes only those properties that any competent speaker of a language would associate with that predicate. These properties are not enough to make true ordinary statements like ‘every dog in my house is asleep’. But we can make sense of the intensional view of terms by considering its origins. One of the founders of the intensional logic tradition is Leibniz, who thinks that all truths are grounded in the nature of individuals. The complete concept of an individual contains everything that is true of it. Building on this, we can see that the complete concept of a term will include enough to ground any truth about it as well.

In both the intensional and extensional logic traditions, we see theories of complex terms. In the extensional tradition, disjunctive and conjunctive terms are interpreted by taking the union and intersection of classes. The conjunctive term AB is interpreted as the intersection of the class A and the class B and the extension of the disjunctive term A+B is understood as the union of the extensions of A and B.

In the intensional tradition, the reverse holds. The term AB is interpreted as the union of the properties in the intent of A and the intent of B and A+B is interpreted as the intersection of the properties in A and B. This reversal makes sense, since more things fit a smaller number of properties and fewer things fit a larger number of properties.

Although some of the logicians working in term logic have very complicated treatments of negation, we can see the origin of the modern conception in the extensional tradition as well. In Boole and most of his followers, the negation of a term is understood as the set theoretic complement of the class represented by that term. For this reason, the negation of classical propositional logic is often called ‘Boolean negation’.

11.3 THE LOGIC OF RELATIVES

Notes

In Charles Peirce's ‘Logic of Relatives’ (1883), we see a move towards an understanding of terms as functions. One problem with traditional term logic is that it lacks the ability to deal with relations. Peirce's logic of relatives is meant to remedy that. He adds terms to Boolean algebra that represent relations, and gives an extensional interpretation of them. They are not propositional functions in the full sense. Peirce's relatives are ‘common names’ that represent classes of pairs of objects (1883, 328). Thus, the logic of relatives represents a generalization of traditional logic rather than a departure from it.

Peirce extends the algebra of terms to deal with particular features of relations. Like other terms, we can have conjunctive, disjunctive, and negative terms. Where f and g are relatives, then fg represents the class of pairs (I,J) such that I bears both f and g to J. Similarly, the disjunctive relative, f+g is such that it represent (I,J) if I bears either f or g to J and f′—the negation of the term f—represents the class of pairs (I,J) such that f does not hold between them. Peirce also has a composition operator, ; , such that f;g names (I,J) if there is some entity K such that f names (I,K) and g names (K,J).

In ‘The Critic of Arguments’ (1892), Peirce adopts a notion that is even closer to that of a propositional function. There he develops the concept of the ‘rhema’. He says the rhema is like a relative term, but it is not a term. It contains a copula, that is, when joined to the correct number of arguments it produces an assertion. For example, ‘__ is bought by __ from __ for __’ is a four-place rhema. Applying it to four objects a, b, c, and d produces the assertion that a is bought by b from c for d (ibid. 420).

One especially interesting point about Peirce's rhema is that he uses the same chemical analogy as Frege does when they discuss the relation between relations and their arguments. They both compare relations (and properties) to ‘atoms or radicals with unsaturated bonds’. What exactly this analogy says of relations or properties, either in Frege or Peirce is somewhat unclear.

11.4 PROPOSITIONAL FUNCTIONS AND THE BIRTH OF MATHEMATICAL LOGIC

In the work of Giuseppe Peano (1858–1932), we find another important step towards the modern notion of a propositional function. Although his work is not as sophisticated as Frege's (see below), it is important because it is influential particularly on Bertrand Russell.

In his ‘Principles of Arithmetic Presented by a New Method’ (1889), Peano introduces propositional connectives in the modern sense (an implication, negation, conjunction, disjunction, and a biconditional) and propositional constants (a verum and a falsum).

More important for us is his treatment of quantification. Peano allows propositions to contain variables, that is to say, he utilizes open formulas. He does not give an interpretation of open formulas. He does not tell us what they represent. But they are used in his theory of quantification. Peano only has a universal quantifier. He does not define an existential quantifier in the ‘Principles’. The quantifier is always attached to a conditional or biconditional. Quantified propositions are always of the form

A ⊃x,y,… B

or

A =x,y,… B

Peano reads ‘A ⊃x,y,… B’ as saying ‘whatever x,y,… may be, from the proposition A one deduces B’ and ‘=’ is Peano's biconditional, that he defines in the usual way from the conditional and conjunction. But he provides us with no more interpretation than that. He refers to variables as ‘indeterminate objects’, but does not discuss what this or what a proposition (or propositional function) that contains propositional objects might be.

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. Discuss the Pre-History.

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2. What do you know the Logic of Relatives?

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3. Discuss the Propositional Functions and the Birth of Mathematical Logic.

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11.5 FREGEAN FUNCTIONS AND CONCEPTS

In Frege we have a fairly general interpretation of sentences as expressing functions applying to arguments. The view that I explore here is one that he develops in the 1890s.

Consider the sentence
My dog is asleep on the floor.

This sentence, like all linguistic expressions, has both a sense and a referent. Its sense is an abstract object—a thought. Its referent is its truth value (which at the moment is the True). We will discuss Frege's analysis of the thought soon, but right now let us look at the referents of the expressions that make up this sentence.
The expression ‘my dog’, according to Frege, is a singular term. It picks out an object (my dog, Zermela). The expression ‘is asleep on the floor’ refers to a concept. Concepts are functions. In this case, the concept is a function from objects to truth values (which are also objects). So, we can treat the above sentence as representing the concept __ is asleep on the floor as applying to the object my dog.

Frege's concepts are very nearly propositional functions in the modern sense. Frege explicitly recognizes them as functions. Like Peirce's rhema, a concept is unsaturated. They are in some sense incomplete. Although Frege never gets beyond the metaphorical in his description of the incompleteness of concepts and other functions, one thing is clear: the distinction between objects and functions is the main division in his metaphysics. There is something special about functions that makes them very different from objects.

Let us consider 'my dog is asleep on the floor' again. Frege thinks that this sentence can be analyzed in various different ways. Instead of treating it as expressing the application of __ is asleep on the floor to my dog, we can think of it as expressing the application of the concept

my dog is asleep on __

to the object

the floor

(see Frege 1919). Frege recognizes what is now a commonplace in the logical analysis of natural language. We can attribute more than one logical form to a single sentence. Let us call this the principle of multiple analyses. Frege does not claim that the principle always holds, but as we shall see, modern type theory does claim this.

With regard to the sense of sentences, they are also the result of applying functions to objects. The sense of 'my dog' is an abstract object. The sense of 'is asleep on the floor' is a function from individual senses, like that of 'my dog', to thoughts (see Frege 1891). The sense of 'is asleep on the floor' is a conceptual sense. It would seem that the principle of multiple analyses holds as much for senses as it does for referents. Frege, however, sometimes talks as if the senses of the constituent expressions of a sentence are actually contained somehow in the thought. It is difficult to understand how all such senses could be in the thought if there are different ways in which the sentence can be analyzed into constituent expressions.

In addition to concepts and conceptual senses, Frege holds that there are extensions of concepts. Frege calls an extension of a concept a 'course of values'. A course of values is determined by the value that the concept has for each of its arguments. Thus, the course of values for the concept __ is a dog records that its value for the argument Zermela is the True and for Socrates is the False, and so on. If two concepts have the same values for every argument, then their courses of values are the same. Thus, courses of values are extensional.

For more about Frege's theory of concepts and its relation to his logic, see the entry on Frege's theorem and foundations for arithmetic.

11.6 THE EMERGENCE OF PROPOSITIONAL FUNCTIONS

The term 'propositional function' appears in print for the first time in Bertrand Russell's Principles of Mathematics (1903). Russell introduces the notion through a discussion of kinds of propositions. Consider propositions of the type that says of something that it is a dog. This is the kind 'x is a dog'. This kind is a propositional function that takes any object o to the proposition that o is a dog.

In this period, Russell holds that propositions are entities that have individuals and properties and relations as constituents. The proposition that Socrates is a man has Socrates and the property of being a man as constituents. In complex propositions the relation between propositional function and the proposition is less clear. Like Frege, Russell allows the abstraction of a propositional function from any omission of an entity from a proposition. Thus, we can view the proposition if Socrates drinks hemlock he will die as representing the application of the function x drinks hemlock ⊃ x will die to Socrates, or the function Socrates will drink x ⊃ Socrates will die to hemlock, and so on. In other words, Russell accepts the principle of multiple analyses.

In the Principles, the quantifier 'all' is analyzed as a part of referring phrases that pick out classes (1903, 72). This, we can see, is a hold-over from the 19th Century extensional logicians (see Section 1). But in slightly later works, such as 'On Denoting' (1905), propositional functions are said to be constituents of universal propositions. According to this analysis the proposition expressed by sentences such as 'All dogs bark' is made up of the propositional function x is a dog ⊃ x barks and a function (of propositional functions) that is represented by the quantifier phrase 'all'. Quantified propositions are interesting for us because they contain propositional functions as constituents.

It is unclear whether Russell holds that propositional functions also occur as constituents in singular propositions like if Socrates drinks hemlock he will die. These propositions do contain properties, like dies, and relations, like drinks, but it is controversial as to whether Russell thinks that these are propositional functions (see Linsky 1999 and Landini 1998).

11.7 PROPOSITIONAL FUNCTIONS IN SIMPLE TYPE THEORY

While writing the Principles of Mathematics, Russell discovered the paradox that now bears his name. Before we get to Russell's paradox, let us discuss some the method of diagonalization by which this and many other paradoxes are generated.

The power set of a set S, ℘S contains all the subsets of S. Georg Cantor (1845–1918) used the method of diagonalization to show that for any set S, ℘S is larger than S.

Here is Cantor's proof. Suppose that ℘S and S are the same size. Then, by the set-theoretic definition of "same size" (more correctly, 'same cardinality') there is a one-to-one surjection between S and ℘S. This means that there is a function that matches up each member of S with a unique member of ℘S so that there are no members of ℘S left over. Let us call this function, f. Then, if x is a member of S, f(x) is in ℘S. Now, since ℘S is the power set of S, it may be that x is in f(x) or it may not be in f(x). Let us now define a set C:

$$C = \{x \in S: x \notin f(x)\}$$

Clearly, C is a subset of S, so it is in ℘S. By hypothesis, f is onto—for every member y of ℘S, there is an x ∈ S such that f(x) = y. Thus there must be some c ∈ S such that

$$f(c) = C$$

Now, either

$c \in C$

or

$c \notin C$.

Suppose that c is in C. Then, by the definition of C, c is not in f(c). That is to say, $c \notin C$. But, if c is not in C, then $c \notin f(c)$. So, by the definition of C, c is in C. Thus, c is in C if and only if c is not in C.

Therefore, the assumption that a set is the same size as its power set leads to a paradox, and so this assumption must be false.

Cantor's theorem has important consequences for the theory of propositional functions. Consider a model for a (first-order) logical language that has a domain D. The variables of the language range over members of D. Now let us add predicate variables to the language. These stand for propositional functions. How are we to interpret them in the model? The standard way of doing so—that is inherited from the extensional logic tradition—is to have predicate variables range over subsets of the domain. A model in which predicate variables range over all subsets of the domain is called a ‘standard model’ for second-order logic. Cantor's theorem tells us that the domain for predicate variables in the standard model is larger than the domain for individual variables. If we have predicates of predicates, then the domain for third order predicates is even larger. And so on.

Russell's paradox is very closely related to Cantor's theorem. There are two versions of the paradox: (1) the class version; (2) the propositional function version. I only discuss the propositional function version of the paradox.

In his early writings, Russell wants logic to be a universal science. It should allow us to talk about properties of everything. By this he means that the variables in logic should be taken to range over all entities. But propositional functions, at least in the Principles, are entities. So variables should range over them. Now consider the predicate R such that,

$$(\forall x)(Rx = \neg xx)$$

(Russell's predicate R is very similar to Cantor's set C.) If we instantiate and substitute R for x, we obtain

$RR \equiv \neg RR$

It seems, then, that the treatment of variables as completely general together with the liberty to define propositional functions by means of any well-formed formula enables us to derive a contradiction.

Russell blocks the contradiction in the Principles by the introduction of a theory of types. This is a simple theory of types, that only distinguishes between the types of various propositional functions (or, in its class-form, of classes). Let us depart from Russell's own exposition of the theory of types in order to give a more rigorous and more modern version of the theory. This will make my presentations of the ramified theory of types and more modern versions of type theory easier.

We'll use one basic type, i (the type of individuals) and define the types as follows:

i is a type; if t1,…, tn are types, then so is <t1,…, tn>, where $n \geq 0$.

Nothing else is a type except by repeated applications of (1) and (2).

The type <t1,…, tn> is the type of a relation among entities of types t1,…, tn. But, for simplicity, we will interpret this as the type of a function that takes these entities to a proposition. (Note that when $n = 0$, then the empty type, < >, is the type for propositions.) This definition incorporates the idea of a well-founded structure. There are no cycles here. We cannot have a function that takes as an argument a function of the same or higher type. Thus, simple type theory bans the sort of self-application that gives rise to Russell's paradox.

The type hierarchy corresponds neatly to the hierarchy of domains that we saw in our discussion of Cantor's theorem. A unary predicate has the type <i>; its domain is D—the set of individuals. A unary predicate of predicates has the type <<i>>, and this corresponds to the domain of subsets of D. And so on.

11.8 PROPOSITIONAL FUNCTIONS IN RAMIFIED TYPE THEORY

After the Principles, however, Russell comes to believe that the simple theory of types is insufficient. The reason for it has to do with the liar paradox. Suppose that 'L' is a name for the proposition:

L is false.

This statement is false if and only if it is true. The problem here has something to do with self-reference, but it cannot be avoided by the simple theory of types alone. For simple types only give us a hierarchy of types of propositional functions. In simple type theory, all propositions have the same type.

The idea behind ramified type theory is to introduce a hierarchy of propositions as well. On this view, propositions and propositional functions have an order. If a propositional function is applied to a proposition of a particular order, then it yields a proposition of a higher order. And every function must have a higher order than its arguments. Thus, we avoid the liar paradox by banning a proposition from occurring within itself. If a proposition p occurs within another proposition, as the argument of a function such as x is false, then the resulting proposition is of a higher order than p.

Unfortunately, Russell never gives a precise formulation of ramified type theory. Perhaps the best formulation is due to Alonzo Church (1976).[1]

Almost at the same time as he adopts the ramified theory of types, Russell abandons propositions. From about 1908 until 1918, although Russell retains the idea that there are true propositions, he denies that there are false ones. When we think about something that is false, say, Zermela is a cat, we are not thinking about a false proposition, but rather the objects of our thought are just Zermela and the property of being a cat. It might seem odd to have a hierarchy especially designed to stratify the propositions and then claim that there are no propositions. Some interpreters, however, have claimed that Russell's denial of the existence of propositions should not be taken seriously and that there are very good reasons to read Principia as being largely a theory of propositions (see Church 1984).

One reason to take the ramified theory of types seriously (even without accepting propositions) is that it can be usefully incorporated into a substitutional theory of quantification. On the substitutional interpretation of the quantifiers, a universally quantified formula such as (∀x)Fx is true if and only if all of its instances Fa1, Fa2, Fa3,… are true. Similarly, (∀x)Fa is true if and only if at least one of its instances is true.

Consider a substitutional interpretation of quantifiers with variables ranging over predicates, as in the formula, (∀P)Pa. This formula is true if and only if all of its instances are true. On a simple theory of types, the type of the variable P is <i>, since its arguments are all individuals (or singular terms). But the simple type of the function, (∀P)Px is also <i>. So an instance of (∀P)Pa is (∀P)Pa itself. A substitutional interpretation of the quantifiers requires that instances be simpler than the formulas of which they are instances. In this case, all we find out is that a particular formula is true only if it is true. This is uninformative and it seems viciously circular.

To block this sort of circularity, we can turn to the ramified theory of types. On the ramified theory, the propositional function (∀P)Px is of order 2, because of the presence of the quantifier binding a variable of order 1. In this way, the ramified theory forces formulas to be simpler (at least in terms of order) than the formulas of which they are instances (see Hazen and Davoren 2000).

11.9 WHAT IS A PROPOSITIONAL FUNCTION IN RUSSELL?

After 1905, we see in Russell a parsimonious inclination. He wants to eliminate entities from his ontology. Some time between 1908 and 1910 he begins to deny the existence of propositions and this denial continues until he develops a theory of propositions as structures of images or words in (1918). What, then, is the fate of propositional functions? It might seem difficult to understand what a propositional function is without the existence of propositions, but Russell's view is, not that complicated. Russell only rejects false propositions. He retains facts in his ontology. Propositional functions, in Principia, are what we now call ‘partial functions’. That is to say, they do not always have values. For example, the propositional function __ is a dog does not have a value for the Sydney Opera House taken as an argument, but it does have a value when my dog is taken as its argument. So, the rejection of false propositions does not cause a serious problem for the theory of propositional functions in Russell.

Having dealt with that problem, let us go on to see what Whitehead and Russell think the nature of propositional functions is. In Principia, they say:

By a 'propositional function' we mean something which contains a variable x, and expresses a proposition as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. (1910, 38).

In this passage, it seems as though they are saying that a propositional function is an ambiguous proposition. In light of the rejection of propositions, this view is especially hard to understand. Urquhart (2003) says that for Whitehead and Russell, a propositional function is something rather like a formula. This seems right, since propositional functions contain variables.

But what exactly are propositional functions in Principia? This is a matter of heated debate among Russell scholars. Perhaps the most influential interpretation is the constructive interpretation, due to Kurt Gödel (1944). On this interpretation, propositional functions are human constructs of some sort. They depend on our ability to think about them or refer to them. A version of the constructive interpretation can also be found in Linsky (1999). There is also a more nominalist interpretation in Landini (1998). On the realist side, are the interpretations given by Alonzo Church (1984) and Warren Goldfarb (1989). Goldfarb thinks that the logical theory of Principia is motivated by Russell's attempt to find the real nature of propositional functions and that this nature is independent of our thinking about it. Goldfarb has a good point, since Russell's logic is supposed to be a perspicuous representation of the way things are. But Russell often seems to deny that propositional functions are real entities.

11.10 POSSIBLE WORLDS AND PROPOSITIONAL FUNCTIONS

Jumping ahead some decades, adding possible worlds together with set theory to the logicians' toolbox has provided them with a very powerful and flexible framework for doing semantics.

First, let us recall the modern notion of a function. A function is a set of ordered pairs. If <a,b> is in a function f, this means that the value of f for the argument a is b or, more concisely, f(a) = b. By the mathematical definition of a function, for each argument of a function there is one and only one value. So, if the ordered pair <a,b> is in a function f and so is <a,c>, then b is the same thing as c.

The construction of propositional functions begins with possible worlds and the assumption that there are sets. Let us call the set of possible worlds W. A proposition is a set of possible worlds. The proposition that Zermela barks, for example, is all the sets of worlds in which Zermela barks. We also need to assume that there is a set I of possible individuals (i.e., the individuals that exist in at least one possible world). We now have all the materials to construct a simple type-theoretic hierarchy of functions.

The usual treatment of the meaning of predicates differs slightly from the manner I have described here. Usually, the intension of a predicate is taken to be a function from possible worlds to sets of individuals (or sets of ordered pairs of individuals for binary relations, ordered triples for three place relations, and so on). Strictly speaking, these functions are not propositional functions because they do not take propositions as values. But for each such function, we can construct an ‘equivalent’ propositional functions by using a process called ‘Currying’ after the logician Haskell Curry. Let's start with a function f from worlds to sets of individuals. Then we can construct the corresponding propositional function g as follows. For each world w and individual i, we construct g so that

w is in g(i) if and only if i is in f(w).

So, the more standard treatment of the meanings of predicates is really equivalent to the use of propositional functions.

11.11 MONTAGUE SEMANTICS

Now that we have a whole hierarchy of propositional functions, we should find some work for them to do. One theory in which propositional functions do good work is Montague semantics, developed in the late 1960s by Richard Montague.

In order to understand Montague's method we need to understand lambda abstraction. For the formula A(x) we read the expression λx[A(x)] as a predicate expression. It extension (in a given possible world) is the set of things that satisfy the formula A(x). Lambda abstractors are governed by two rules, known as α-conversion and β-reduction:

(α-con) A(a) (a formula with a free for x) can be replaced by λx[A(x)]a.

(β-red) λx[A(x)]a can be replaced by A(a) (where x is free for a in A(x)).

Because of the equivalence between a formula A(x) and λx[A(x)]a, one might wonder why add lambda abstractors to our language. In Montague semantics, the answer has to do with the very direct way that he translates expressions of natural languages into his logical language. We will discuss that soon, but first let us learn a bit about Montague's intensional logic.

Montague adds two other pieces of notation to his language: ∧ and ∨. The expression ∧λx[Fx] represents a function from worlds to sets of individuals. Given a possible world w, ∧λx[Fx] represents a function that takes w to the extension of λx[Fx]. The operator ∨ takes expressions of the form ∧λx[Fx] 'down' to their extensions at the world in which the expression is being evaluated. For example, the extension of ∨∧λx[Fx] at w is just the same as the extension of λx[Fx] at w.

What is so special about Montague semantics is that it can be used in a very direct way as semantics for large fragments of natural languages. Consider the following sentence:

Zermela barks.

The meaning of this sentence is understood in Montague semantics as a structure of the meanings of its constituent expressions. Montague represents the meanings of expressions using translation rules. Here we use the following translation rules:

Zermela translates into $\lambda P[(\vee P)z]$

barks translates into $\wedge B$

Now we can construct a formula that gives the meaning of ‘Zermela barks’:

$\lambda P[(\vee P)z]\wedge B$

Notice that in constructing the sentence we place the expressions in the same order in which they occur in English. The use of lambda abstracts allows us to reverse the order of two expressions from the way in which they would appear in ordinary statements of a formal logical language (that does not have lambdas). Now we can use β-reduction to obtain:

$(\vee\wedge B)z$

And now we apply Montague's rule to eliminate $\vee\wedge$:

Bz

In this process we start with an expression that has the same order of expressions as the original English sentence and then reduce it to a very standard formula of logic. This tells us that the truth condition of the sentence ‘Zermela barks’ is the set of worlds that is the proposition expressed by Bz. Of course we knew that independently of Montague's work, but the point is that the Montague reduction shows us how we can connect the surface grammar of English sentences to the formula of our logical language. The formula of standard logic, moreover, displays its truth-conditions in a very perspicuous way. So, the Montague reduction shows us the connection between sentences of natural languages to their truth conditions.

11.12 CATEGORIAL GRAMMAR

Categorial grammars were first constructed in the 1930s by Kazamir Ajdukiewicz (1890–1963), and developed by Yehoshua Bar Hillel

(1915–1975) and Joachim Lambek (1922–) in the 1950s an 1960s. Categorial grammars are logical tools for representing the syntax of languages.

In categorial grammar, the syntax of languages is represented using a different sort of generalization of the functional notation than in Montague semantics. In Montague Semantics, the lambda abstractor is used to move the meaning of an expression to the location that the expression occupies in a sentence. In categorial grammar, predicates and many other sorts of expressions are taken to be functions of sorts. But there is a distinction in categorial grammar between two sorts of application of a function to its arguments.

Let's see how this works. Let's start with the primitive types CN (common noun) and NP (noun phrase). The indefinite article 'a' takes a common noun (on its right) and returns a NP. So it has the type NP/CN. The common noun 'dog', of course, has the type CN. We write 'A has the type T' as 'A⊢T'. So we have,

a ⊢ NP/CN

and

dog ⊢ CN

In order to put these two sequents together, we can use a form of the rule modus ponens which says that from a sequent X ⊢ A/B and a sequent Y ⊢ B, we can derive the sequent X.Y ⊢ A. We can use this rule to derive:

a.dog ⊢ NP

Moreover, an intransitive verb has the type NP\S, where S is the type of sentences. The backslash in NP\S means that the expression takes an argument of type NP on the left side and returns an expression of type S. The verb 'barks' is intransitive, that is,

barks ⊢ NP\S

The version of modus ponens that we use with the backslash is slightly different. It tells us that from X ⊢ A\B and Y ⊢ A we can derive Y.X ⊢ B. So we now can obtain,

(a.dog).barks ⊢ S

This says that ‘a dog barks’ is a sentence.

The logics used to describe grammars in this way are substructural logics.

What is of interest to us here is that in categorial grammars determiners such as ‘a’ and verbs are thought of as functions, but they can differ from one another in terms of whether they take arguments on their right or on their left. In the set theoretic concept of function as a set of ordered pairs, functions are thought of just in terms of their correlating arguments with values. A function, as it is understood in categorial grammar has more structure than this. This is an interesting generalization of the notion of a function as it is used in logic. We can see that it also has important links to the concept of a propositional function, especially as it is used in Montague semantics.

In categorial grammar we can attribute more than one type to a single expression in a language. Let us call this the principle of multiple types. Here is an example due to Mark Steadman. Consider the sentence

I dislike, and Mary enjoys musicals.

The transitive verbs ‘dislike’ and ‘enjoys’ have the type (NP\S)/NP, that is, they take a noun phrase on their right and return a verb phrase. But in the case of ‘I dislike, and Mary enjoys musicals’ the verbs are separated from their object and joined to their objects. Steadman deals with this by raising the type of the subjects ‘I’ and ‘Mary’. Usually, we treat these words as having the type NP, but here they have the type S/(NP\S). This is the type of an expression that takes a verb phrase on its right and returns a sentence. Steadman then uses a rule that makes the backslash transitive and derives that ‘I.dislike’ has the type S/NP, which takes a noun phrase (such as ‘musicals’) on its right an returns a sentence.

We can see that the principle of multiple types also holds if analyze sentences other type theories, such as the simple theory of types. For consider the sentence

Mary eats a hamburger.

In interpreting this sentence we can take ‘Mary’ to be of type i, but we can also take it to be of type <<i>>, that is, the type of a propositional function on propositional functions of individuals. We can also raise the type of ‘eats a hamburger’ to <<<i>>>, a propositional function on propositional functions on propositional functions on individuals. And so on. The principle of multiple types and the principle of multiple analyses together show that a single expression or sentence can be interpreted as having a very large number of logical forms.

Check Your Progress 2

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. Discuss about the Fregean Functions and Concepts.

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2. Describe The Emergence of Propositional Functions.

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3. What do you know about the Propositional Functions in Simple Type Theory?

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11.13 LET US SUM UP

This brief history of propositional functions shows that they are useful entities and that they have played a central role in logic as it is used in philosophy and linguistics. I have omitted the more mathematical uses of propositional functions, for example, in Russell's and Ramsey's constructions of classes, and in treatments of general models for higher-order logic. But the topic of propositional functions is a big one and we can't cover it all in a single encyclopedia article.

11.14 KEY WORDS

Propositional: The term proposition has a broad use in contemporary analytic philosophy. The most basic meaning is a statement proposing an idea that can be true or false.

Semantics: Semantics is the linguistic and philosophical study of meaning in language, programming languages, formal logics, and semiotics. It is concerned with the relationship between signifiers—like words, phrases, signs, and symbols—and what they stand for in reality, their denotation.

11.15 QUESTIONS FOR REVIEW

1. Discuss the Propositional Functions in Ramified Type Theory
2. What is a Propositional Function in Russell?
3. What do you know about the Possible Worlds and Propositional Functions?
4. Discuss the Montague Semantics.
5. Discuss the Categorial Grammar.

11.16 SUGGESTED READINGS AND REFERENCES

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- Goldblatt, Robert, 2011, Quantifers, Propositions and Identity, Cambridge: Cambridge University Press. (This presents a new semantics for modal predicate logic that uses propositions as well as worlds. Chapter 4 explores some formal reasons for also adding propositional functions to the semantics.)
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- Whitehead, Alfred North, and Bertrand Russell, 1910–1913 [1925], Principia Mathematica, Cambridge: Cambridge University Press. (This is a sustained, but extremely difficult, presentation of ramified type theory.)

11.17 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. See Section 11.2
2. See Section 11.3
3. See Section 11.4

Check Your Progress 2

1. See Section 11.5
2. See Section 11.6
3. See Section 11.7

UNIT 12: CLASSICAL LOGIC: SOME NORMAL PREPOSITIONAL MODAL SYSTEM

STRUCTURE

12.0 OBJECTIVES

After this unit, we can able to know:

- To know the importance of Language in Modal logic.
- To discuss the Deduction
- To discuss Semantics
- To know the Meta-theory
- To discuss the One Right Logic?

12.1 INTRODUCTION

Normal Propositional Modal Logics

Propositional modal logics are formed from classical propositional logic by adding two new (interdefinable) sentence operators: ("necessarily", sometimes rendered as L) and ♦ (“possibly”, sometimes rendered as M). These are unary operators: they operate on a single sentence (which of course might be complex, and might even contain other occurrences of the operators). Here’s a definition of (most of) the well-known normal modal systems, described by the method of Chellas (1980) of starting with system K, and adding various axioms to it that describe the more complex systems. Thus the modal system KD45 results from adding axioms D, 4, and 5 to system K. As it turns out, some axioms imply others, some combinations of axioms are equivalent to each other, and some combinations are known by other names. When using axiom systems, normal modal logics are built upon system K, which is: 1. Classical propositional logic (however you wish to present it) 2. \`k (p ≡ ¬ϕ¬p) [interdefinability of and ϕ] 3. \`k (p ⊃ q) ⊃ (p ⊃ q) [the K-axiom] 4. if \`k p then \`k p [the rule of necessitation, N] 5. if \`k p and \`k (p ⊃ q) then \`k q [Modus Ponens] Now consider the following six axioms: D. p ⊃ ϕp T. p ⊃ p G. ϕp ⊃ ϕp B. p ⊃ ϕp 4. p ⊃ p 5. ϕp ⊃ ϕp Starting with K (which adds 0 of these axioms), there are 26 (=64) different combinations of the six axioms. However, there are certain implications between axioms and equivalences amongst groups of axioms, so we do not get 64 different modal systems. The relevant implications are: T implies D B implies G 5 implies G and the following equivalences KB4 is equivalent to KB5 KDB4, KTB4, KT45, KT5, KTB5 are equivalent to one another. (And any other implications this yields). This leaves us with 21 modal systems. They are listed and diagrammed on the document “ModalLogicDiagram”, which is elsewhere on this course page. Normal modal systems have a semantics described by a binary accessibility relation (Rxy) on a set of “possible worlds” using the definitions of truth for modal statements:

Typically, logic consists of a formal or informal language together with a deductive system and/or a model-theoretic semantics. The language has components that correspond to a part of a natural language like English or Greek. The deductive system is to capture, codify, or simply record arguments that are valid for the given language, and the semantics is to capture, codify, or record the meanings, or truth-conditions for at least part of the language.

The following sections provide the basics of a typical logic, sometimes called “classical elementary logic” or “classical first-order logic”. Section 2 develops a formal language, with a rigorous syntax and grammar. The formal language is a recursively defined collection of strings on a fixed alphabet. As such, it has no meaning, or perhaps better, the meaning of its formulas is given by the deductive system and the semantics. Some of the symbols have counterparts in ordinary language. We define an argument to be a non-empty collection of sentences in the formal language, one of which is designated to be the conclusion. The other sentences (if any) in an argument are its premises. Section 3 sets up a deductive system for the language, in the spirit of natural deduction. An argument is derivable if there is a deduction from some or all of its premises to its conclusion. Section 4 provides a model-theoretic semantics. An argument is valid if there is no interpretation (in the semantics) in which its premises are all true and its conclusion false. This reflects the longstanding view that a valid argument is truth-preserving.

Thus, deductions preserve truth. Then we establish a converse, called completeness, that an argument is valid only if it is derivable. This establishes that the deductive system is rich enough to provide a deduction for every valid argument. So there are enough deductions: all and only valid arguments are derivable. We briefly indicate other features of the logic, some of which are corollaries to soundness and completeness.

Today, logic is a branch of mathematics and a branch of philosophy. In most large universities, both departments offer courses in logic, and there is usually a lot of overlap between them. Formal languages, deductive systems, and model-theoretic semantics are mathematical objects and, as such, the logician is interested in their mathematical properties and relations. Soundness, completeness, and most of the other results reported below are typical examples. Philosophically, logic is at least closely related to the study of correct reasoning. Reasoning is an epistemic, mental activity. So logic is at least closely allied with epistemology. Logic is also a central branch of computer science, due, in part, to interesting computational relations in logical systems, and, in part, to the close connection between formal deductive argumentation and reasoning (see the entries on recursive functions, computability and complexity, and philosophy of computer science).

This raises questions concerning the philosophical relevance of the various mathematical aspects of logic. How do deducibility and validity, as properties of formal languages--sets of strings on a fixed alphabet--relate to correct reasoning? What do the mathematical results reported below have to do with the original philosophical issues concerning valid reasoning? This is an instance of the philosophical problem of explaining how mathematics applies to non-mathematical reality.

Typically, ordinary deductive reasoning takes place in a natural language, or perhaps a natural language augmented with some mathematical symbols. So our question begins with the relationship between a natural language and a formal language. Without attempting to be comprehensive, it may help to sketch several options on this matter.

One view is that the formal languages accurately exhibit actual features of certain fragments of a natural language. Some philosophers claim that declarative sentences of natural language have underlying logical forms and that these forms are displayed by formulas of a formal language. Other writers hold that (successful) declarative sentences express propositions; and formulas of formal languages somehow display the forms of these propositions. On views like this, the components of logic provide the underlying deep structure of correct reasoning. A chunk of reasoning in natural language is correct if the forms underlying the sentences constitute a valid or deducible argument. See for example, Montague [1974], Davidson [1984], Lycan [1984] (and the entry on logical form).

Another view, held at least in part by Gottlob Frege and Wilhelm Leibniz, is that because natural languages are fraught with vagueness and ambiguity, they should be replaced by formal languages. A similar view, held by W. V. O. Quine (e.g., [1960], [1986]), is that a natural language should be regimented, cleaned up for serious scientific and metaphysical work. One desideratum of the enterprise is that the logical structures in the regimented language should be transparent. It should be easy to "read off" the logical properties of each sentence. A regimented language is similar to a formal language regarding, for example, the explicitly presented rigor of its syntax and its truth conditions.

On a view like this, deducibility and validity represent idealizations of correct reasoning in natural language. A chunk of reasoning is correct to the extent that it corresponds to, or can be regimented by, a valid or deducible argument in a formal language.

When mathematicians and many philosophers engage in deductive reasoning, they occasionally invoke formulas in a formal language to help disambiguate, or otherwise clarify what they mean. In other words, sometimes formulas in a formal language are used in ordinary reasoning. This suggests that one might think of a formal language as an addendum to a natural language. Then our present question concerns the relationship between this addendum and the original language. What do deducibility and validity, as sharply defined on the addendum, tell us about correct deductive reasoning in general?

Another view is that a formal language is a mathematical model of a natural language in roughly the same sense as, say, a collection of point masses is a model of a system of physical objects, and the Bohr construction is a model of an atom. In other words, a formal language displays certain features of natural languages, or idealizations thereof, while ignoring or simplifying other features. The purpose of mathematical models is to shed light on what they are models of, without claiming that the model is accurate in all respects or that the model should replace what it is a model of. On a view like this, deducibility and validity represent mathematical models of (perhaps different aspects of) correct reasoning in natural languages. Correct chunks of deductive reasoning correspond, more or less, to valid or deducible arguments; incorrect chunks of reasoning roughly correspond to invalid or non-deducible arguments. See, for example, Corcoran [1973], Shapiro [1998], and Cook [2002].

12.2 LANGUAGE

Here we develop the basics of a formal language, or to be precise, a class of formal languages. Again, a formal language is a recursively defined set of strings on a fixed alphabet. Some aspects of the formal languages correspond to, or have counterparts in, natural languages like English. Technically, this "counterpart relation" is not part of the formal development, but we will mention it from time to time, to motivate some of the features and results.

12.2.1 Building blocks

We begin with analogues of singular terms, linguistic items whose function is to denote a person or object. We call these terms. We assume a stock of individual constants. These are lower-case letters, near the beginning of the Roman alphabet, with or without numerical subscripts:
a,a1,b23,c,d22,etc.a,a1,b23,c,d22,etc.
We envisage a potential infinity of individual constants. In the present system each constant is a single character, and so individual constants do not have an internal syntax. Thus we have an infinite alphabet. This could be avoided by taking a constant like d22d22, for example, to consist of three characters, a lowercase "dd" followed by a pair of subscript "2"s.
We also assume a stock of individual variables. These are lower-case letters, near the end of the alphabet, with or without numerical subscripts:
w,x,y12,z,z4,etc.w,x,y12,z,z4,etc.
In ordinary mathematical reasoning, there are two functions terms need to fulfill. We need to be able to denote specific, but unspecified (or arbitrary) objects, and sometimes we need to express generality. In our system, we use some constants in the role of unspecified reference and variables to express generality. Both uses are recapitulated in the formal treatment below. Some logicians employ different symbols for unspecified objects (sometimes called "individual parameters") and variables used to express generality.
Constants and variables are the only terms in our formal language, so all of our terms are simple, corresponding to proper names and some uses of pronouns. We call a term closed if it contains no variables. In general, we use vv to represent variables, and tt to represent a closed term. Some authors also introduce function letters, which allow complex terms corresponding to: "7+47+4" and "the wife of Bill Clinton", or complex terms containing variables, like "the father of xx" and "x/yx/y". Logic

books aimed at mathematicians are likely to contain function letters, probably due to the centrality of functions in mathematical discourse. Books aimed at a more general audience (or at philosophy students), may leave out function letters, since it simplifies the syntax and theory. We follow the latter route here. This is an instance of a general tradeoff between presenting a system with greater expressive resources, at the cost of making its formal treatment more complex.

For each natural number nn, we introduce a stock of nn-place predicate letters. These are upper-case letters at the beginning or middle of the alphabet. A superscript indicates the number of places, and there may or may not be a subscript. For example,

A3,B32,P3,etc.A3,B23,P3,etc.

are three-place predicate letters. We often omit the superscript, when no confusion will result. We also add a special two-place predicate symbol "==" for identity.

Zero-place predicate letters are sometimes called "sentence letters". They correspond to free-standing sentences whose internal structure does not matter. One-place predicate letters, called "monadic predicate letters", correspond to linguistic items denoting properties, like "being a man", "being red", or "being a prime number". Two-place predicate letters, called "binary predicate letters", correspond to linguistic items denoting binary relations, like "is a parent of" or "is greater than". Three-place predicate letters correspond to three-place relations, like "lies on a straight line between". And so on.

The non-logical terminology of the language consists of its individual constants and predicate letters. The symbol "==", for identity, is not a non-logical symbol. In taking identity to be logical, we provide explicit treatment for it in the deductive system and in the model-theoretic semantics. Most authors do the same, but there is some controversy over the issue (Quine [1986, Chapter 5]). If KK is a set of constants and predicate letters, then we give the fundamentals of a language L1K=L1K= built on this set of non-logical terminology. It may be called the first-order language with identity on KK. A similar language that lacks the symbol for identity (or which takes identity to be non-logical) may be called L1KL1K, the first-order language without identity on KK.

12.2.2 Atomic formulas

If VV is an nn-place predicate letter in KK, and t1,…,tnt1,…,tn are terms of KK, then Vt1…tnVt1…tn is an atomic formula of L1K=L1K=. Notice that the terms t1,…,tnt1,…,tn need not be distinct. Examples of atomic formulas include:

P4xaab,C1x,C1a,D0,A3abc.P4xaab,C1x,C1a,D0,A3abc.

The last one is an analogue of a statement that a certain relation (A)(A) holds between three objects (a,b,c)(a,b,c). If t1t1 and t2t2 are terms, then t1=t2t1=t2 is also an atomic formula of L1K=L1K=. It corresponds to an assertion that t1t1 is identical to t2t2. If an atomic formula has no variables, then it is called an atomic sentence. If it does have variables, it is called open. In the above list of examples, the first and second are open; the rest are sentences.

12.2.3 Compound formulas

We now introduce the final items of the lexicon:

¬,&,∨,→,∀,∃,(,)

We give a recursive definition of a formula of L1K=:

All atomic formulas of L1K= are formulas of L1K=.

If θ is a formula of L1K=, then so is ¬θ.

A formula corresponding to ¬θ thus says that it is not the case that θ. The symbol "¬" is called "negation", and is a unary connective.

If θ and ψ are formulas of L1K=, then so is (θ&ψ).

The ampersand "&" corresponds to the English "and" (when "and" is used to connect sentences). So (θ&ψ) can be read "θ and ψ". The formula (θ&ψ) is called the "conjunction" of θ and ψ.

If θ and ψ are formulas of L1K=, then so is (θ∨ψ).

The wedge "∨" corresponds to "either … or … or both", so (θ∨ψ) can be read "θ or ψ". The formula (θ∨ψ) is called the "disjunction" of θ and ψ.

5. If θ and ψ are formulas of L1K=, then so is (θ→ψ).
The arrow "→" roughly corresponds to "if … then … ", so (θ→ψ) can be read "if θ then ψ" or "θ only if ψ".

The symbols "&", "V", and "→" are called "binary connectives", since they serve to "connect" two formulas into one. Some authors introduce (θ↔ψ) as an abbreviation of ((θ→ψ)&(ψ→θ)). The symbol "↔" is an analogue of the locution "if and only if".
If θ is a formula of L1K= and v is a variable, then ∀vθ is a formula of L1K=.
The symbol "∀" is called a universal quantifier, and is an analogue of "for all"; so ∀vθ can be read "for all v,θ".

If θ is a formula of L1K= and v is a variable, then ∃vθ is a formula of L1K=.
The symbol "∃" is called an existential quantifier, and is an analogue of "there exists" or "there is"; so ∃vθ can be read "there is a v such that θ".

That's all folks. That is, all formulas are constructed in accordance with rules (1)–(7).
Clause (8) allows us to do inductions on the complexity of formulas. If a certain property holds of the atomic formulas and is closed under the operations presented in clauses (2)–(7), then the property holds of all formulas. Here is a simple example:

Theorem 1. Every formula of L1K= has the same number of left and right parentheses. Moreover, each left parenthesis corresponds to a unique right parenthesis, which occurs to the right of the left parenthesis. Similarly, each right parenthesis corresponds to a unique left parenthesis, which occurs to the left of the given right parenthesis. If a parenthesis occurs between a matched pair of parentheses, then its mate also occurs within that matched pair. In other words, parentheses that occur within a matched pair are themselves matched.

Proof: By clause (8), every formula is built up from the atomic formulas using clauses (2)–(7). The atomic formulas have no parentheses. Parentheses are introduced only in clauses (3)–(5), and each time they are introduced as a matched set. So at any stage in the construction of a formula, the parentheses are paired off.

We next define the notion of an occurrence of a variable being free or bound in a formula. A variable that immediately follows a quantifier (as in "∀x" and "∃y") is neither free nor bound. We do not even think of those as occurrences of the variable. All variables that occur in an atomic formula are free. If a variable occurs free (or bound) in θ or in ψ, then that same occurrence is free (or bound) in ¬θ,(θ&ψ),(θ∨ψ), and (θ→ψ). That is, the (unary and binary) connectives do not change the status of variables that occur in them. All occurrences of the variable v in θ are bound in ∀vθ and ∃vθ. Any free occurrences of v in θ are bound by the initial quantifier. All other variables that occur in θ are free or bound in ∀vθ and ∃vθ, as they are in θ.

For example, in the formula (∀x(Axy ∨Bx)&Bx), the occurrences of "x" in Axy and in the first Bx are bound by the quantifier. The occurrence of "y" and last occurrence of "x" are free. In ∀x(Ax→∃xBx), the "x" in Ax is bound by the initial universal quantifier, while the other occurrence of x is bound by the existential quantifier. The above syntax allows this "double-binding". Although it does not create any ambiguities (see below), we will avoid such formulas, as a matter of taste and clarity.

The syntax also allows so-called vacuous binding, as in ∀xBc. These, too, will be avoided in what follows. Some treatments of logic rule out vacuous binding and double binding as a matter of syntax. That simplifies some of the treatments below, and complicates others.

Free variables correspond to place-holders, while bound variables are used to express generality. If a formula has no free variables, then it is called a sentence. If a formula has free variables, it is called open.

12.2.4 Features of the syntax

Before turning to the deductive system and semantics, we mention a few features of the language, as developed so far. This helps draw the contrast between formal languages and natural languages like English.

We assume at the outset that all of the categories are disjoint. For example, no connective is also a quantifier or a variable, and the non-logical terms are not also parentheses or connectives. Also, the items within each category are distinct. For example, the sign for disjunction does not do double-duty as the negation symbol, and perhaps more significantly, no two-place predicate is also a one-place predicate.

One difference between natural languages like English and formal languages like L1K= is that the latter are not supposed to have any ambiguities. The policy that the different categories of symbols do not overlap, and that no symbol does double-duty, avoids the kind of ambiguity, sometimes called "equivocation", that occurs when a single word has two meanings: "I'll meet you at the bank." But there are other kinds of ambiguity. Consider the English sentence:

John is married, and Mary is single, or Joe is crazy.

It can mean that John is married and either Mary is single or Joe is crazy, or else it can mean that either both John is married and Mary is single, or else Joe is crazy. An ambiguity like this, due to different ways to parse the same sentence, is sometimes called an "amphiboly". If our formal language did not have the parentheses in it, it would have amphibolies. For example, there would be a "formula" A&B∨ C. Is this supposed to be ((A&B)∨C), or is it (A&(B∨C))? The parentheses resolve what would be an amphiboly.

Can we be sure that there are no other amphibolies in our language? That is, can we be sure that each formula of L1K= can be put together in only one way? Our next task is to answer this question.

Let us temporarily use the term "unary marker" for the negation symbol (¬) or a quantifier followed by a variable (e.g., ∀x,∃z).

Lemma . Each formula consists of a string of zero or more unary markers followed by either an atomic formula or a formula produced using a binary connective, via one of clauses (3)–(5).

Proof: We proceed by induction on the complexity of the formula or, in other words, on the number of formation rules that are applied. The

Lemma clearly holds for atomic formulas. Let n be a natural number, and suppose that the Lemma holds for any formula constructed from n or fewer instances of clauses (2)–(7). Let θ be a formula constructed from n+1 instances. The Lemma holds if the last clause used to construct θ was either (3), (4), or (5). If the last clause used to construct θ was (2), then θ is ¬ψ. Since ψ was constructed with n instances of the rule, the Lemma holds for ψ (by the induction hypothesis), and so it holds for θ. Similar reasoning shows the Lemma to hold for θ if the last clause was (6) or (7). By clause (8), this exhausts the cases, and so the Lemma holds for θ, by induction.

Lemma 3. If a formula θ contains a left parenthesis, then it ends with a right parenthesis, which matches the leftmost left parenthesis in θ.
Proof: Here we also proceed by induction on the number of instances of (2)–(7) used to construct the formula. Clearly, the Lemma holds for atomic formulas, since they have no parentheses. Suppose, then, that the Lemma holds for formulas constructed with n or fewer instances of (2)–(7), and let θ be constructed with n+1 instances. If the last clause applied was (3)–(5), then the Lemma holds since θ itself begins with a left parenthesis and ends with the matching right parenthesis. If the last clause applied was (2), then θ is ¬ψ, and the induction hypothesis applies to ψ. Similarly, if the last clause applied was (6) or (7), then θ consists of a quantifier, a variable, and a formula to which we can apply the induction hypothesis. It follows that the Lemma holds for θ.

Lemma 4. Each formula contains at least one atomic formula.

The proof proceeds by induction on the number of instances of (2)–(7) used to construct the formula, and we leave it as an exercise.

Theorem 5. Let α,β be nonempty sequences of characters on our alphabet, such that αβ (i.e α followed by β) is a formula. Then α is not a formula.
Proof: By Theorem 1 and Lemma 3, if α contains a left parenthesis, then the right parenthesis that matches the leftmost left parenthesis in αβ

comes at the end of αβ, and so the matching right parenthesis is in β. So, α has more left parentheses than right parentheses. By Theorem 1,α is not a formula. So now suppose that α does not contain any left parentheses. By Lemma 2,αβ consists of a string of zero or more unary markers followed by either an atomic formula or a formula produced using a binary connective, via one of clauses (3)–(5). If the latter formula was produced via one of clauses (3)–(5), then it begins with a left parenthesis. Since α does not contain any parentheses, it must be a string of unary markers. But then α does not contain any atomic formulas, and so by Lemma 4,α is not a formula. The only case left is where αβ consists of a string of unary markers followed by an atomic formula, either in the form t1=t2 or Pt1…tn. Again, if α just consisted of unary markers, it would not be a formula, and so α must consist of the unary markers that start αβ, followed by either t1 by itself, t1= by itself, or the predicate letter P, and perhaps some (but not all) of the terms t1,…,tn. In the first two cases, α does not contain an atomic formula, by the policy that the categories do not overlap. Since P is an n-place predicate letter, by the policy that the predicate letters are distinct, P is not an m-place predicate letter for any m≠n. So the part of α that consists of P followed by the terms is not an atomic formula. In all of these cases, then, α does not contain an atomic formula. By Lemma 4,α is not a formula.

We are finally in position to show that there is no amphiboly in our language.

Theorem 6. Let θ be any formula of L1K=. If θ is not atomic, then there is one and only one among (2)–(7) that was the last clause applied to construct θ. That is, θ could not be produced by two different clauses. Moreover, no formula produced by clauses (2)–(7) is atomic.

Proof: By Clause (8), either θ is atomic or it was produced by one of clauses (2)–(7). Thus, the first symbol in θ must be either a predicate letter, a term, a unary marker, or a left parenthesis. If the first symbol in θ is a predicate letter or term, then θ is atomic. In this case, θ was not produced by any of (2)–(7), since all such formulas begin with something other than a predicate letter or term. If the first symbol in θ is a negation sign “¬”, then was θ produced by clause (2), and not by any other clause (since the other clauses produce formulas that begin with either a quantifier or a left parenthesis). Similarly, if θ begins with a universal quantifier, then it was produced by clause (6), and not by any other clause, and if θ begins with an existential quantifier, then it was produced by clause (7), and not by any other clause. The only case left is where θ begins with a left parenthesis. In this case, it must have been produced by one of (3)–(5), and not by any other clause. We only need to rule out the possibility that θ was produced by more than one of (3)–(5). To take an example, suppose that θ was produced by (3) and (4). Then θ is (ψ1&ψ2) and θ is also (ψ3∨ψ4), where ψ1,ψ2,ψ3, and ψ4 are themselves formulas. That is, (ψ1&ψ2) is the very same formula as (ψ3∨ψ4). By Theorem 5,ψ1 cannot be a proper part of ψ3, nor can ψ3 be a proper part of ψ1. So ψ1 must be the same formula as ψ3. But then “&” must be the same symbol as “∨”, and this contradicts the policy that all of the symbols are different. So θ was not produced by both Clause (3) and Clause (4). Similar reasoning takes care of the other combinations.

This result is sometimes called “unique readability”. It shows that each formula is produced from the atomic formulas via the various clauses in exactly one way. If θ was produced by clause (2), then its main connective is the initial “¬”. If θ was produced by clauses (3), (4), or (5), then its main connective is the introduced “&”, “∨”, or “→”, respectively. If θ was produced by clauses (6) or (7), then its main connective is the initial quantifier. We apologize for the tedious details. We included them to indicate the level of precision and rigor for the syntax.

12.3 DEDUCTION

We now introduce a deductive system, D, for our languages. As above, we define an argument to be a non-empty collection of sentences in the formal language, one of which is designated to be the conclusion. If there are any other sentences in the argument, they are its premises.[1] By convention, we use “Γ”, “Γ′”, “Γ1”, etc, to range over sets of formulas, and we use the letters “ϕ”, “ψ”, “θ”, uppercase or lowercase, with or without subscripts, to range over single formulas. We write "Γ,Γ′" for the union of Γ and Γ′, and "Γ,ϕ" for the union of Γ with {ϕ}.

We write an argument in the form ⟨Γ,ϕ⟩, where Γ is a set of sentences, the premises, and ϕ is a single sentence, the conclusion. Remember that Γ may be empty. We write Γ⊢ϕ to indicate that ϕ is deducible from Γ, or, in other words, that the argument ⟨Γ,ϕ⟩ is deducible in D. We may write Γ⊢Dϕ to emphasize the deductive system D. We write ⊢ϕ or ⊢Dϕ to indicate that ϕ can be deduced (in D) from the empty set of premises.

The rules in D are chosen to match logical relations concerning the English analogues of the logical terminology in the language. Again, we define the deducibility relation by recursion. We start with a rule of assumptions:

(As) If ϕ is a member of Γ, then Γ⊢ϕ.

We thus have that {ϕ}⊢ϕ; each premise follows from itself. We next present two clauses for each connective and quantifier. The clauses indicate how to "introduce" and "eliminate" sentences in which each symbol is the main connective.

First, recall that "&" is an analogue of the English connective "and". Intuitively, one can deduce a sentence in the form (θ&ψ) if one has deduced θ and one has deduced ψ. Conversely, one can deduce θ from (θ&ψ) and one can deduce ψ from (θ&ψ):

(&I) If Γ1⊢θ and Γ2⊢ψ, then Γ1,Γ2⊢(θ&ψ).

(&E) If Γ⊢(θ&ψ) then Γ⊢θ; and if Γ⊢(θ&ψ) then Γ⊢ψ.

The name "&I" stands for "&-introduction"; "&E" stands for "&-elimination".

Since, the symbol "∨" corresponds to the English "or", (θ∨ψ) should be deducible from θ, and (θ∨ψ) should also be deducible from ψ:

(∨I) If Γ⊢θ then Γ⊢(θ∨ψ); if Γ⊢ψ then Γ⊢(θ∨ψ).

The elimination rule is a bit more complicated. Suppose that "θ or ψ" is true. Suppose also that ϕ follows from θ and that ϕ follows from ψ. One can reason that if θ is true, then ϕ is true. If instead ψ is true, we still have that ϕ is true. So either way, ϕ must be true.

(∨E) If $\Gamma 1 \vdash (\theta \vee \psi), \Gamma 2, \theta \vdash \phi$ and $\Gamma 3, \psi \vdash \phi$, then $\Gamma 1, \Gamma 2, \Gamma 3 \vdash \phi$.

For the next clauses, recall that the symbol, "→", is an analogue of the English "if … then … " construction. If one knows, or assumes $(\theta \rightarrow \psi)$ and also knows, or assumes θ, then one can conclude ψ. Conversely, if one deduces ψ from an assumption θ, then one can conclude that $(\theta \rightarrow \psi)$.

(→I) If $\Gamma, \theta \vdash \psi$, then $\Gamma \vdash (\theta \rightarrow \psi)$.

(→E) If $\Gamma 1 \vdash (\theta \rightarrow \psi)$ and $\Gamma 2 \vdash \theta$, then $\Gamma 1, \Gamma 2 \vdash \psi$.

This elimination rule is sometimes called "modus ponens". In some logic texts, the introduction rule is proved as a "deduction theorem".

Our next clauses are for the negation sign, "¬". The underlying idea is that a sentence ψ is inconsistent with its negation $\neg\psi$. They cannot both be true. We call a pair of sentences $\psi, \neg\psi$ contradictory opposites. If one can deduce such a pair from an assumption θ, then one can conclude that θ is false, or, in other words, one can conclude $\neg\theta$.

(¬I) If $\Gamma 1, \theta \vdash \psi$ and $\Gamma 2, \theta \vdash \neg\psi$, then $\Gamma 1, \Gamma 2 \vdash \neg\theta$.

By (As), we have that $\{A, \neg A\} \vdash A$ and $\{A, \neg A\} \vdash \neg A$. So by ¬I we have that $\{A\} \vdash \neg\neg A$. However, we do not have the converse yet. Intuitively, $\neg\neg\theta$ corresponds to "it is not the case that it is not the case that" . One might think that this last is equivalent to θ, and we have a rule to that effect:

(DNE) If $\Gamma \vdash \neg\neg\theta$, then $\Gamma \vdash \theta$.

The name DNE stands for "double-negation elimination". There is some controversy over this inference. It is rejected by philosophers and mathematicians who do not hold that each meaningful sentence is either true or not true. Intuitionistic logic does not sanction the inference in question (see, for example Dummett [2000], or the entry on intuitionistic logic, or history of intuitionistic logic), but, again, classical logic does.

To illustrate the parts of the deductive system D presented thus far, we show that $\vdash (A \vee \neg A)$:

$\{\neg(A\lor\neg A),A\}\vdash\neg(A\lor\neg A)$, by (As)

$\{\neg(A\lor\neg A),A\}\vdash A$, by (As).

$\{\neg(A\lor\neg A),A\}\vdash(A\lor\neg A)$, by ($\lor$I), from (ii).

$\{\neg(A\lor\neg A)\}\vdash\neg A$, by ($\neg$I), from (i) and (iii).

$\{\neg(A\lor\neg A),\neg A\}\vdash\neg(A\lor\neg A)$, by (As)

$\{\neg(A\lor\neg A),\neg A\}\vdash\neg A$, by (As)

$\{\neg(A\lor\neg A),\neg A\}\vdash(A\lor\neg A)$, by ($\lor$I), from (vi).

$\{\neg(A\lor\neg A)\}\vdash\neg\neg A$, by ($\neg$I), from (v) and (vii).

$\vdash\neg\neg(A\lor\neg A)$, by ($\neg$I), from (iv) and (viii).

$\vdash(A\lor\neg A)$, by (DNE), from (ix).

The principle $(\theta\lor\neg\theta)$ is sometimes called the law of excluded middle. It is not valid in intuitionistic logic.

Let $\theta,\neg\theta$ be a pair of contradictory opposites, and let ψ be any sentence at all. By (As) we have $\{\theta,\neg\theta,\neg\psi\}\vdash\theta$ and $\{\theta,\neg\theta,\neg\psi\}\vdash\neg\theta$. So by ($\neg$I), $\{\theta,\neg\theta\}\vdash\neg\neg\psi$. So, by (DNE) we have $\{\theta,\neg\theta\}\vdash\psi$. That is, anything at all follows from a pair of contradictory opposites. Some logicians introduce a rule to codify a similar inference:

If $\Gamma1\vdash\theta$ and $\Gamma2\vdash\neg\theta$, then for any sentence $\psi,\Gamma1,\Gamma2\vdash\psi$

The inference is sometimes called ex falso quodlibet or, more colorfully, explosion. Some call it "¬-elimination", but perhaps this stretches the notion of "elimination" a bit. We do not officially include ex falso quodlibet as a separate rule in D, but as will be shown below (Theorem 10), each instance of it is derivable in our system D.

Some logicians object to ex falso quodlibet, on the ground that the sentence ψ may be irrelevant to any of the premises in Γ. Suppose, for example, that one starts with some premises Γ about human nature and facts about certain people, and then deduces both the sentence "Clinton had extra-marital sexual relations" and "Clinton did not have extra-marital sexual relations". One can perhaps conclude that there is something wrong with the premises Γ. But should we be allowed to then deduce anything at all from Γ? Should we be allowed to deduce "The economy is sound"?

A small minority of logicians, called dialetheists, hold that some contradictions are actually true. For them, ex falso quodlibet is not truth-preserving.

Deductive systems that demur from ex falso quodlibet are called paraconsistent. Most relevant logics are paraconsistent. See the entries on relevance logic, paraconsistent logic, and dialetheism. Or see Anderson and Belnap [1975], Anderson, Belnap, and Dunn [1992], and Tennant [1997] for fuller overviews of relevant logic; and Priest [2006],[2006a] for dialetheism. Deep philosophical issues concerning the nature of logical consequence are involved. Far be it for an article in a philosophy encyclopedia to avoid philosophical issues, but space considerations preclude a fuller treatment of this issue here. Suffice it to note that the inference ex falso quodlibet is sanctioned in systems of classical logic, the subject of this article. It is essential to establishing the balance between the deductive system and the semantics (see §5 below).

The next pieces of D are the clauses for the quantifiers. Let θ be a formula, v a variable, and t a term (i.e., a variable or a constant). Then define θ(v|t) to be the result of substituting t for each free occurrence of v in θ. So, if θ is (Qx&∃xPxy), then θ(x|c) is (Qc&∃xPxy). The last occurrence of x is not free.

A sentence in the form ∀vθ is an analogue of the English "for every v,θ holds". So one should be able to infer θ(v|t) from ∀vθ for any closed term t. Recall that the only closed terms in our system are constants.

(∀E) If Γ⊢∀vθ, then Γ⊢θ(v|t), for any closed term t.

The idea here is that if ∀vθ is true, then θ should hold of t, no matter what t is.

The introduction clause for the universal quantifier is a bit more complicated. Suppose that a sentence θ contains a closed term t, and that θ has been deduced from a set of premises Γ. If the closed term t does not occur in any member of Γ, then θ will hold no matter which object t may denote. That is, ∀vθ follows.

(∀I) For any closed term t, if $\Gamma \vdash \theta(v|t)$, then $\Gamma \vdash \forall v\theta$ provided that t is not in Γ or θ.

This rule (∀I) corresponds to a common inference in mathematics. Suppose that a mathematician says "let n be a natural number" and goes on to show that n has a certain property P, without assuming anything about n (except that it is a natural number). She then reminds the reader that n is "arbitrary", and concludes that P holds for all natural numbers. The condition that the term t not occur in any premise is what guarantees that it is indeed "arbitrary". It could be any object, and so anything we conclude about it holds for all objects.

The existential quantifier is an analogue of the English expression "there exists", or perhaps just "there is". If we have established (or assumed) that a given object t has a given property, then it follows that there is something that has that property.

(∃I) For any closed term t, if $\Gamma \vdash \theta(v|t)$ then $\Gamma \vdash \exists v\theta$.

The elimination rule for ∃ is not quite as simple:

(∃E) For any closed term t, if $\Gamma1 \vdash \exists v\theta$ and $\Gamma2, \theta(v|t) \vdash \phi$, then $\Gamma1, \Gamma2 \vdash \phi$, provided that t does not occur in ϕ, Γ2 or θ.

This elimination rule also corresponds to a common inference. Suppose that a mathematician assumes or somehow concludes that there is a natural number with a given property P. She then says "let n be such a natural number, so that Pn", and goes on to establish a sentence ϕ, which does not mention the number n. If the derivation of ϕ does not invoke anything about n (other than the assumption that it has the given property P), then n could have been any number that has the property P. That is, n is an arbitrary number with property P (this is where we invoke constants which "denote" arbitrary objects). It does not matter which number n is. Since ϕ does not mention n, it follows from the assertion that something has property P. The provisions added to (∃E) are to guarantee that t is "arbitrary".

The final items are the rules for the identity sign "=". The introduction rule is about a simple as can be:

(=I) $\Gamma \vdash t=t$, where t is any closed term.

This "inference" corresponds to the truism that everything is identical to itself. The elimination rule corresponds to a principle that if a is identical to b, then anything true of a is also true of b.

(=E) For any closed terms t1 and t2, if $\Gamma 1 \vdash t1=t2$ and $\Gamma 2 \vdash \theta$, then $\Gamma 1, \Gamma 2 \vdash \theta'$, where θ' is obtained from θ by replacing one or more occurances of t1 with t2.

The rule (=E) indicates a certain restriction in the expressive resources of our language. Suppose, for example, that Harry is identical to Donald (since his mischievous parents gave him two names). According to most people's intuitions, it would not follow from this and "Dick knows that Harry is wicked" that "Dick knows that Donald is wicked", for the reason that Dick might not know that Harry is identical to Donald. Contexts like this, in which identicals cannot safely be substituted for each other, are called "opaque". We assume that our language L1K= has no opaque contexts.

One final clause completes the description of the deductive system D:

(*) That's all folks. $\Gamma \vdash \theta$ only if θ follows from members of Γ by the above rules.

Again, this clause allows proofs by induction on the rules used to establish an argument. If a property of arguments holds of all instances of (As) and (=I), and if the other rules preserve the property, then every argument that is deducible in D enjoys the property in question.

Before moving on to the model theory for L1K=, we pause to note a few features of the deductive system. To illustrate the level of rigor, we begin with a lemma that if a sentence does not contain a particular closed term, we can make small changes to the set of sentences we prove it from without problems. We allow ourselves the liberty here of extending some previous notation: for any terms t and t', and any formula θ, we say that $\theta(t|t')$ is the result of replacing all free occurrences of t in θ with t'.

Lemma 7. If $\Gamma 1$ and $\Gamma 2$ differ only in that wherever $\Gamma 1$ contains θ, $\Gamma 2$ contains $\theta(t|t')$, then for any sentence ϕ not containing t or t′, if $\Gamma 1 \vdash \phi$ then $\Gamma 2 \vdash \phi$.

Proof: The proof proceeds by induction on the number of steps in the proof of ϕ. Crucial to this proof is the fact that $\theta = \theta(t|t')$ whenever θ does not contain t or t′. When the number of steps in the proof of ϕ is one, this means that the last (and only) rule applied is (As) or (=I). Then, since ϕ does not contain t or t′, if $\Gamma 1 \vdash \phi$ we simply apply the same rule ((As) or (=I)) to $\Gamma 2$ to get $\Gamma 2 \vdash \phi$. Assume that there are $n>1$ steps in the proof of ϕ, and that Lemma 7 holds for any proof with less than n steps. Suppose that the nth rule applied to $\Gamma 1$ was (&I). Then ϕ is $\psi \& \chi$, and $\Gamma 1 \vdash \phi \& \chi$. But then we know that previous steps in the proof include $\Gamma 1 \vdash \psi$ and $\Gamma 1 \vdash \chi$, and by induction, we have $\Gamma 2 \vdash \psi$ and $\Gamma 2 \vdash \chi$, since neither ψ nor χ contain t or t′. So, we simply apply (&I) to $\Gamma 2$ to get $\Gamma 2 \vdash \psi \& \chi$ as required. Suppose now that the last step applied in the proof of $\Gamma 1 \vdash \phi$ was (&E). Then, at a previous step in the proof of ϕ, we know $\Gamma 1 \vdash \phi \& \psi$ for some sentence ψ. If ψ does not contain t, then we simply apply (&E) to $\Gamma 2$ to obtain the desired result. The only complication is if ψ contains t. Then we would have that $\Gamma 2 \vdash (\phi \& \psi)(t|t')$. But, since $(\phi \& \psi)(t|t')$ is $\phi(t|t') \& \psi(t|t')$, and $\phi(t|t')$ is just ϕ, we can just apply (&E) to get $\Gamma 2 \vdash \phi$ as required. The cases for the other rules are similar.

Theorem 8. The rule of Weakening. If $\Gamma 1 \vdash \phi$ and $\Gamma 1 \subseteq \Gamma 2$, then $\Gamma 2 \vdash \phi$.

Proof: Again, we proceed by induction on the number of rules that were used to arrive at $\Gamma 1 \vdash \phi$. Suppose that $n>0$ is a natural number, and that the theorem holds for any argument that was derived using fewer than n rules. Suppose that $\Gamma 1 \vdash \phi$ using exactly n rules. If $n=1$, then the rule is either (As) or (=I). In these cases, $\Gamma 2 \vdash \phi$ by the same rule. If the last rule applied was (&I), then ϕ has the form $(\theta \& \psi)$, and we have $\Gamma 3 \vdash \theta$ and $\Gamma 4 \vdash \psi$, with $\Gamma 1 = \Gamma 3, \Gamma 4$. We apply the induction hypothesis to the deductions of θ and ψ, to get $\Gamma 2 \vdash \theta$ and $\Gamma 2 \vdash \psi$. and then apply (&I) to the result to get $\Gamma 2 \vdash \phi$. Most of the other cases are exactly like this. Slight complications arise only in the rules (∀I) and (∃E), because there we have to pay attention to the conditions for the rules.

Suppose that the last rule applied to get $\Gamma 1 \vdash \phi$ is ($\forall$I). So ϕ is a sentence of the form $\forall v\theta$, and we have $\Gamma 1 \vdash \theta(v|t)$ and t does occur in any member of $\Gamma 1$ or in θ. The problem is that t may occur in a member of $\Gamma 2$, and so we cannot just invoke the induction hypothesis and apply ($\forall$I) to the result. So, let t′ be a term not occurring in any sentence in $\Gamma 2$. Let Γ' be the result of substituting t′ for all t in $\Gamma 2$. Then, since t does not occur in $\Gamma 1$, $\Gamma 1 \subseteq \Gamma'$. So, the induction hypothesis gives us $\Gamma' \vdash \theta(v|t)$, and we know that Γ' does not contain t, so we can apply ($\forall$I) to get $\Gamma' \vdash \forall v\theta$. But $\forall v\theta$ does not contain t or t′, so $\Gamma 2 \vdash \forall v\theta$ by Lemma 7.

Suppose that the last rule applied was ($\exists$E), we have $\Gamma 3 \vdash \exists v\theta$ and $\Gamma 4, \theta(v|t) \vdash \phi$, with $\Gamma 1$ being $\Gamma 3, \Gamma 4$, and t not in ϕ, $\Gamma 4$ or θ. If t does not occur free in $\Gamma 2$, we apply the induction hypothesis to get $\Gamma 2 \vdash \exists v\theta$, and then ($\exists$E) to end up with $\Gamma 2 \vdash \phi$. If t does occur free in $\Gamma 2$, then we follow a similar proceedure to $\forall$I, using Lemma 7.

Theorem 8 allows us to add on premises at will. It follows that $\Gamma \vdash \phi$ if and only if there is a subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$. Some systems of relevant logic do not have weakening, nor does substructural logic (See the entries on relevance logic, substructural logics, and linear logic).

By clause (*), all derivations are established in a finite number of steps. So we have

Theorem 9. $\Gamma \vdash \phi$ if and only if there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$.

Theorem 10. The rule of ex falso quodlibet is a “derived rule” of D: if $\Gamma 1 \vdash \theta$ and $\Gamma 2 \vdash \neg\theta$, then $\Gamma 1, \Gamma 2 \vdash \psi$, for any sentence ψ.

Proof: Suppose that $\Gamma 1 \vdash \theta$ and $\Gamma 2 \vdash \neg\theta$. Then by Theorem 8,$\Gamma 1, \neg\psi \vdash \theta$, and $\Gamma 2, \neg\psi \vdash \neg\theta$. So by ($\neg$I), $\Gamma 1, \Gamma 2 \vdash \neg\neg\psi$. By (DNE), $\Gamma 1, \Gamma 2 \vdash \psi$.

Theorem 11. The rule of Cut. If $\Gamma 1 \vdash \psi$ and $\Gamma 2, \psi \vdash \theta$, then $\Gamma 1, \Gamma 2 \vdash \theta$.

Proof: Suppose $\Gamma 1 \vdash \psi$ and $\Gamma 2, \psi \vdash \theta$. We proceed by induction on the number of rules used to establish $\Gamma 2, \psi \vdash \theta$. Suppose that n is a natural number, and that the theorem holds for any argument that was derived using fewer than n rules. Suppose that $\Gamma 2,\psi \vdash \theta$ was derived using exactly n rules. If the last rule used was (=I), then $\Gamma 1,\Gamma 2 \vdash \theta$ is also an instance of (=I). If $\Gamma 2,\psi \vdash \theta$ is an instance of (As), then either θ is ψ, or θ is a member of $\Gamma 2$. In the former case, we have $\Gamma 1 \vdash \theta$ by supposition, and get $\Gamma 1,\Gamma 2 \vdash \theta$ by Weakening (Theorem 8). In the latter case, $\Gamma 1,\Gamma 2 \vdash \theta$ is itself an instance of (As). Suppose that $\Gamma 2,\psi \vdash \theta$ was obtained using (&E). Then we have $\Gamma 2,\psi \vdash (\theta \& \phi)$. The induction hypothesis gives us $\Gamma 1,\Gamma 2 \vdash (\theta \& \phi)$, and (&E) produces $\Gamma 1,\Gamma 2 \vdash \theta$. The remaining cases are similar.

Theorem 11 allows us to chain together inferences. This fits the practice of establishing theorems and lemmas and then using those theorems and lemmas later, at will. The cut principle is, some think, essential to reasoning. In some logical systems, the cut principle is a deep theorem; in others it is invalid. The system here was designed, in part, to make the proof of Theorem 11 straightforward.

If $\Gamma \vdash D\theta$, then we say that the sentence θ is a deductive consequence of the set of sentences Γ, and that the argument $\langle \Gamma,\theta \rangle$ is deductively valid. A sentence θ is a logical theorem, or a deductive logical truth, if $\vdash D\theta$. That is, θ is a logical theorem if it is a deductive consequence of the empty set. A set Γ of sentences is consistent if there is no sentence θ such that $\Gamma \vdash D\theta$ and $\Gamma \vdash D\neg\theta$. That is, a set is consistent if it does not entail a pair of contradictory opposite sentencess.

Theorem 12. A set Γ is consistent if and only if there is a sentence θ such that it is not the case that $\Gamma \vdash \theta$.

Proof: Suppose that Γ is consistent and let θ be any sentence. Then either it is not the case that $\Gamma \vdash \theta$ or it is not the case that $\Gamma \vdash \neg\theta$. For the converse, suppose that Γ is inconsistent and let ψ be any sentence. We have that there is a sentence such that both $\Gamma \vdash \theta$ and $\Gamma \vdash \neg\theta$. By ex falso quodlibet (Theorem 10), $\Gamma \vdash \psi$.

Define a set Γ of sentences of the language L1K= to be maximally consistent if Γ is consistent and for every sentence θ of L1K=, if θ is not in Γ, then Γ,θ is inconsistent. In other words, Γ is maximally consistent if Γ is consistent, and adding any sentence in the language not already in Γ

renders it inconsistent. Notice that if Γ is maximally consistent then $\Gamma\vdash\theta$ if and only if θ is in Γ.

Theorem 13. The Lindenbaum Lemma. Let Γ be any consistent set of sentences of L1K=. Then there is a set Γ' of sentences of L1K= such that $\Gamma\subseteq\Gamma'$ and Γ' is maximally consistent.

Proof: Although this theorem holds in general, we assume here that the set K of non-logical terminology is either finite or denumerably infinite (i.e., the size of the natural numbers, usually called $\aleph 0$). It follows that there is an enumeration $\theta 0,\theta 1,\ldots$ of the sentences of L1K=, such that every sentence of L1K= eventually occurs in the list. Define a sequence of sets of sentences, by recursion, as follows: $\Gamma 0$ is Γ; for each natural number n, if $\Gamma n,\theta n$ is consistent, then let $\Gamma n+1=\Gamma n,\theta n$. Otherwise, let $\Gamma n+1=\Gamma n$. Let Γ' be the union of all of the sets Γn. Intuitively, the idea is to go through the sentences of L1K=, throwing each one into Γ' if doing so produces a consistent set. Notice that each Γn is consistent. Suppose that Γ' is inconsistent. Then there is a sentence θ such that $\Gamma'\vdash\theta$ and $\Gamma'\vdash\neg\theta$. By Theorem 9 and Weakening (Theorem 8), there is finite subset Γ'' of Γ' such that $\Gamma''\vdash\theta$ and $\Gamma''\vdash\neg\theta$. Because Γ'' is finite, there is a natural number n such that every member of Γ'' is in Γn. So, by Weakening again, $\Gamma n\vdash\theta$ and $\Gamma n\vdash\neg\theta$. So Γn is inconsistent, which contradicts the construction. So Γ' is consistent. Now suppose that a sentence θ is not in Γ'. We have to show that Γ',θ is inconsistent. The sentence θ must occur in the aforementioned list of sentences; say that θ is θm. Since θm is not in Γ', then it is not in $\Gamma m+1$. This happens only if $\Gamma m,\theta m$ is inconsistent. So a pair of contradictory opposites can be deduced from $\Gamma m,\theta m$. By Weakening, a pair of contradictory opposites can be deduced from $\Gamma',\theta m$. So $\Gamma',\theta m$ is inconsistent. Thus, Γ' is maximally consistent.

Notice that this proof uses a principle corresponding to the law of excluded middle. In the construction of Γ', we assumed that, at each stage, either Γn is consistent or it is not. Intuitionists, who demur from excluded middle, do not accept the Lindenbaum lemma.

12.4 SEMANTICS

Let K be a set of non-logical terminology. An interpretation for the language L1K= is a structure M=⟨d,I⟩, where d is a non-empty set, called the domain-of-discourse, or simply the domain, of the interpretation, and I is an interpretation function. Informally, the domain is what we interpret the language L1K= to be about. It is what the variables range over. The interpretation function assigns appropriate extensions to the non-logical terms. In particular,

If c is a constant in K, then I(c) is a member of the domain d.

Thus we assume that every constant denotes something. Systems where this is not assumed are called free logics (see the entry on free logic). Continuing,

If P0 is a zero-place predicate letter in K, then I(P) is a truth value, either truth or falsehood.

If Q1 is a one-place predicate letter in K, then I(Q) is a subset of d. Intuitively, I(Q) is the set of members of the domain that the predicate Q holds of. If Q represents "red", then I(Q) is the set of red members of the domain.

If R2 is a two-place predicate letter in K, then I(R) is a set of ordered pairs of members of d. Intuitively, I(R) is the set of pairs of members of the domain that the relation R holds between. If R represents "love", then I(R) is the set of pairs ⟨a,b⟩ such that a and b are the members of the domain for which a loves b.

In general, if Sn is an n-place predicate letter in K, then I(S) is a set of ordered n-tuples of members of d.

Define s to be a variable-assignment, or simply an assignment, on an interpretation M, if s is a function from the variables to the domain d of M. The role of variable-assignments is to assign denotations to the free variables of open formulas. (In a sense, the quantifiers determine the "meaning" of the bound variables.)

Let t be a term of L1K=. We define the denotation of t in M under s, in terms of the interpretation function and variable-assignment:

If t is a constant, then DM,s(t) is I(t), and if t is a variable, then DM,s(t) is s(t).

That is, the interpretation M assigns denotations to the constants, while the variable-assignment assigns denotations to the (free) variables. If the language contained function symbols, the denotation function would be defined by recursion.

We now define a relation of satisfaction between interpretations, variable-assignments, and formulas of L1K=. If ϕ is a formula of L1K=,M is an interpretation for L1K=, and s is a variable-assignment on M, then we write M,s⊨ϕ for M satisfies ϕ under the assignment s. The idea is that M,s⊨ϕ is an analogue of "ϕ comes out true when interpreted as in M via s".

We proceed by recursion on the complexity of the formulas of L1K=.

If t1 and t2 are terms, then M,s⊨t1=t2 if and only if DM,s(t1) is the same as DM,s(t2).

This is about as straightforward as it gets. An identity t1=t2 comes out true if and only if the terms t1 and t2 denote the same thing.

If P0 is a zero-place predicate letter in K, then M,s⊨P if and only if I(P) is truth.

If Sn is an n-place predicate letter in K and t1,…,tn are terms, then M,s⊨St1…tn if and only if the n-tuple ⟨DM,s(t1),…,DM,s(tn)⟩ is in I(S).

This takes care of the atomic formulas. We now proceed to the compound formulas of the language, more or less following the meanings of the English counterparts of the logical terminology.

$M,s \vDash \neg\theta$ if and only if it is not the case that $M,s \vDash \theta$.

$M,s \vDash (\theta \& \psi)$ if and only if both $M,s \vDash \theta$ and $M,s \vDash \psi$.

$M,s \vDash (\theta \vee \psi)$ if and only if either $M,s \vDash \theta$ or $M,s \vDash \psi$.

$M,s \vDash (\theta \rightarrow \psi)$ if and only if either it is not the case that $M,s \vDash \theta$, or $M,s \vDash \psi$.

$M,s \vDash \forall v\theta$ if and only if $M,s' \vDash \theta$, for every assignment s' that agrees with s except possibly at the variable v.

The idea here is that $\forall v\theta$ comes out true if and only if θ comes out true no matter what is assigned to the variable v. The final clause is similar.

$M,s \vDash \exists v\theta$ if and only if $M,s' \vDash \theta$, for some assignment s' that agrees with s except possibly at the variable v.

So $\exists v\theta$ comes out true if there is an assignment to v that makes θ true.

Theorem 6, unique readability, assures us that this definition is coherent. At each stage in breaking down a formula, there is exactly one clause to be applied, and so we never get contradictory verdicts concerning satisfaction.
As indicated, the role of variable-assignments is to give denotations to the free variables. We now show that variable-assignments play no other role.

Theorem 14. For any formula θ, if s1 and s2 agree on the free variables in θ, then $M,s1 \vDash \theta$ if and only if $M,s2 \vDash \theta$.

Proof: We proceed by induction on the complexity of the formula θ. The theorem clearly holds if θ is atomic, since in those cases only the values of the variable-assignments at the variables in θ figure in the definition. Assume, then, that the theorem holds for all formulas less complex than

θ. And suppose that s1 and s2 agree on the free variables of θ. Assume, first, that θ is a negation, $\neg\psi$. Then, by the induction hypothesis, $M,s1\vDash\psi$ if and only if $M,s2\vDash\psi$. So, by the clause for negation, $M,s1\vDash\neg\psi$ if and only if $M,s2\vDash\neg\psi$. The cases where the main connective in θ is binary are also straightforward. Suppose that θ is $\exists v\psi$, and that $M,s1\vDash\exists v\psi$. Then there is an assignment $s'1$ that agrees with s1 except possibly at v such that $M,s'1\vDash\psi$. Let $s'2$ be the assignment that agrees with s2 on the free variables not in ψ and agrees with $s'1$ on the others. Then, by the induction hypothesis, $M,s'2\vDash\psi$. Notice that $s'2$ agrees with s2 on every variable except possibly v. So $M,s2\vDash\exists v\psi$. The converse is the same, and the case where θ begins with a universal quantifier is similar.

By Theorem 14, if θ is a sentence, and s1,s2, are any two variable-assignments, then $M,s1\vDash\theta$ if and only if $M,s2\vDash\theta$. So we can just write $M\vDash\theta$ if $M,s\vDash\theta$ for some, or all, variable-assignments s. So we define

$M\vDash\theta$ where θ is a sentence just in case $M,s\vDash\theta$ for all variable assignments s.

In this case, we call M a model of θ.

Suppose that $K'\subseteq K$ are two sets of non-logical terms. If $M=\langle d,I\rangle$ is an interpretation of L1K=, then we define the restriction of M to L1K′ be the interpretation $M'=\langle d,I'\rangle$ such that I′ is the restriction of I to K′. That is, M and M′ have the same domain and agree on the non-logical terminology in K′. A straightforward induction establishes the following:

Theorem 15. If M′ is the restriction of M to L1K′, then for every sentence θ of L1K′, $M\vDash\theta$ if and only if $M'\vDash\theta$.

Theorem 16. If two interpretations M1 and M2 have the same domain and agree on all of the non-logical terminology of a sentence θ, then $M1\vDash\theta$ if and only if $M2\vDash\theta$.

In short, the satisfaction of a sentence θ only depends on the domain of discourse and the interpretation of the non-logical terminology in θ.

We say that an argument $\langle\Gamma,\theta\rangle$ is semantically valid, or just valid, written $\Gamma\vDash\theta$, if for every interpretation M of the language, if $M\vDash\psi$, for every member ψ of Γ, then $M\vDash\theta$. If $\Gamma\vDash\theta$, we also say that θ is a logical consequence, or semantic consequence, or model-theoretic consequence of Γ. The definition corresponds to the informal idea that an argument is valid if it is not possible for its premises to all be true and its conclusion false. Our definition of logical consequence also sanctions the common thesis that a valid argument is truth-preserving--to the extent that satisfaction represents truth. Officially, an argument in L1K= is valid if its conclusion comes out true under every interpretation of the language in which the premises are true. Validity is the model-theoretic counterpart to deducibility.

A sentence θ is logically true, or valid, if $M\vDash\theta$, for every interpretation M. A sentence is logically true if and only if it is a consequence of the empty set. If θ is logically true, then for any set Γ of sentences, $\Gamma\vDash\theta$. Logical truth is the model-theoretic counterpart of theoremhood.

A sentence θ is satisfiable if there is an interpretation M such that $M\vDash\theta$. That is, θ is satisfiable if there is an interpretation that satisfies it. A set Γ of sentences is satisfiable if there is an interpretation M such that $M\vDash\theta$, for every sentence θ in Γ. If Γ is a set of sentences and if $M\vDash\theta$ for each sentence θ in Γ, then we say that M is a model of Γ. So a set of sentences is satisfiable if it has a model. Satisfiability is the model-theoretic counterpart to consistency.

Notice that $\Gamma\vDash\theta$ if and only if the set $\Gamma,\neg\theta$ is not satisfiable. It follows that if a set Γ is not satisfiable, then if θ is any sentence, $\Gamma\vDash\theta$. This is a model-theoretic counterpart to ex falso quodlibet (see Theorem 10). We have the following, as an analogue to Theorem 12:

Theorem 17. Let Γ be a set of sentences. The following are equivalent: (a) Γ is satisfiable; (b) there is no sentence θ such that both $\Gamma\vDash\theta$ and $\Gamma\vDash\neg\theta$; (c) there is some sentence ψ such that it is not the case that $\Gamma\vDash\psi$.

Proof: (a)⇒(b): Suppose that Γ is satisfiable and let θ be any sentence. There is an interpretation M such that $M \vDash \psi$ for every member ψ of Γ. By the clause for negations, we cannot have both $M \vDash \theta$ and $M \vDash \neg\theta$. So either $\langle \Gamma, \theta \rangle$ is not valid or else $\langle \Gamma, \neg\theta \rangle$ is not valid. (b)⇒(c): This is immediate. (c)⇒(a): Suppose that it is not the case that $\Gamma \vDash \psi$. Then there is an interpretation M such that $M \vDash \theta$, for every sentence θ in Γ and it is not the case that $M \vDash \psi$. A fortiori, M satisfies every member of Γ, and so Γ is satisfiable.

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. What do you know the importance of Language in Modal logic?

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2. Discuss the Deduction.

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3. Discuss Semantics.

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12.5 META-THEORY

We now present some results that relate the deductive notions to their model-theoretic counterparts. The first one is probably the most straightforward. We motivated both the various rules of the deductive system D and the various clauses in the definition of satisfaction in terms of the meaning of the English counterparts to the logical terminology (more or less, with the same simplifications in both cases). So one would expect that an argument is deducible, or deductively valid, only if it is semantically valid.

Theorem 18. Soundness. For any sentence θ and set Γ of sentences, if Γ⊢Dθ, then Γ⊨θ.

Proof: We proceed by induction on the number of clauses used to establish Γ⊢θ. So let n be a natural number, and assume that the theorem holds for any argument established as deductively valid with fewer than n steps. And suppose that Γ⊢θ was established using exactly n steps. If the last rule applied was (=I) then θ is a sentence in the form t=t, and so θ is logically true. A fortiori, Γ⊨θ. If the last rule applied was (As), then θ is a member of Γ, and so of course any interpretation that satisfies every member of Γ also satisfies θ. Suppose the last rule applied is (&I). So θ has the form (ϕ&ψ), and we have Γ1⊢ϕ and Γ2⊢ψ, with Γ=Γ1,Γ2. The induction hypothesis gives us Γ1⊨ϕ and Γ2⊨ψ. Suppose that M satisfies every member of Γ. Then M satisfies every member of Γ1, and so M satisfies ϕ. Similarly, M satisfies every member of Γ2, and so M satisfies ψ. Thus, by the clause for "&" in the definition of satisfaction, M satisfies θ. So Γ⊨θ.

Suppose the last clause applied was (∃E). So we have Γ1⊢∃vϕ and Γ2,ϕ(v|t)⊢θ, where Γ=Γ1,Γ2, and t does not occur in ϕ,θ, or in any member of Γ2.

We need to show that Γ⊨θ. By the induction hypothesis, we have that Γ1⊨∃vϕ and Γ2,ϕ(v|t)⊨θ. Let M be an interpretation such that M makes every member of Γ true. So, M makes every member of Γ1 and Γ2 true. Then M,s⊨∃vϕ for all variable assignments s, so there is an s′ such that M,s′⊨ϕ. Let M′ differ from M only in that IM′(t)=s′(v). Then, M′,s′⊨ϕ(v|t) and M′,s′⊨Γ2 since t does not occur in ϕ or Γ2. So, M′,s′⊨θ. Since t does not occur in θ and M′ differs from M only with respect to IM′(t), M,s′⊨θ. Since θ is a sentence, s′ doesn't matter, so M⊨θ as desired. Notice the role of the restrictions on (∃E) here. The other cases are about as straightforward.

Corollary 19. Let Γ be a set of sentences. If Γ is satisfiable, then Γ is consistent.

Proof: Suppose that Γ is satisfiable. So let M be an interpretation such that M satisfies every member of Γ. Assume that Γ is inconsistent. Then there is a sentence θ such that Γ⊢θ and Γ⊢¬θ. By soundness (Theorem 18), Γ⊨θ and Γ⊨¬θ. So we have that M⊨θ and M⊨¬θ. But this is impossible, given the clause for negation in the definition of satisfaction.
Even though the deductive system D and the model-theoretic semantics were developed with the meanings of the logical terminology in mind, one should not automatically expect the converse to soundness (or Corollary 19) to hold. For all we know so far, we may not have included enough rules of inference to deduce every valid argument. The converses to soundness and Corollary 19 are among the most important and influential results in mathematical logic. We begin with the latter.

Theorem 20. Completeness. Gödel [1930]. Let Γ be a set of sentences. If Γ is consistent, then Γ is satisfiable.

Proof: The proof of completeness is rather complex. We only sketch it here. Let Γ be a consistent set of sentences of L1K=. Again, we assume for simplicity that the set K of non-logical terminology is either finite or countably infinite (although the theorem holds even if K is uncountable). The task at hand is to find an interpretation M such that M satisfies every member of Γ. Consider the language obtained from L1K= by adding a denumerably infinite stock of new individual constants c0,c1,… We stipulate that the constants, c0,c1,…, are all different from each other and none of them occur in K. One interesting feature of this construction, due to Leon Henkin, is that we build an interpretation of the language from the language itself, using some of the constants as members of the domain of discourse. Let θ0(x),θ1(x),… be an enumeration of the formulas of the expanded language with at most one free variable, so that each formula with at most one free variable occurs in the list eventually. Define a sequence Γ0,Γ1,… of sets of sentences (of the expanded language) by recursion as follows: Γ0=Γ; and

$\Gamma n+1=\Gamma n,(\exists x\theta n\rightarrow\theta n(x|ci))$, where ci is the first constant in the above list that does not occur in θn or in any member of Γn. The underlying idea here is that if ∃xθnis true, then ci is to be one such x. Let Γ be the union of the sets Γn.

We sketch a proof that Γ′ is consistent. Suppose that Γ′ is inconsistent. By Theorem 9, there is a finite subset of Γ that is inconsistent, and so one of the sets Γm is inconsistent. By hypothesis, Γ0=Γ is consistent. Let n be the smallest number such that Γn is consistent, but $\Gamma n+1=\Gamma n,(\exists x\theta n\rightarrow\theta n(x|ci))$ is inconsistent. By (¬I), we have that

$\Gamma n\vdash\neg(\exists x\theta n\rightarrow\theta n(x|ci))$.(1)

By ex falso quodlibet (Theorem 10), $\Gamma n,\neg\exists x\theta n,\exists x\theta n\vdash\theta n(x|ci)$. So by (→I), $\Gamma n,\neg\exists x\theta n\vdash(\exists x\theta n\rightarrow\theta n(x|ci))$. From this and (1), we have $\Gamma n\vdash\neg\neg\exists x\theta n$, by (¬I), and by (DNE) we have

$\Gamma n\vdash\exists x\theta n$.(2)

By (As), $\Gamma n,\theta n(x|ci),\exists x\theta n\vdash\theta n(x|ci)$. So by (→I), $\Gamma n,\theta n(x|ci)\vdash(\exists x\theta n\rightarrow\theta n(x|ci))$. From this and (1), we have $\Gamma n\vdash\neg\theta n(x|ci)$, by (¬I). Let t be a term that does not occur in θn or in any member of Γn. By uniform substitution of t for ci, we can turn the derivation of $\Gamma n\vdash\neg\theta n(x|ci)$ into $\Gamma n\vdash\neg\theta n(x|t)$. By (∀I), we have

$\Gamma n\vdash\forall v\neg\theta n(x|v)$.(3)

By (As) we have $\{\forall v\neg\theta n(x|v),\theta n\}\vdash\theta n$ and by (∀E) we have $\{\forall v\neg\theta n(x|v),\theta n\}\vdash\neg\theta n$. So $\{\forall v\neg\theta n(x|v),\theta n\}$ is inconsistent. Let ϕ be any sentence of the language. By ex falso quodlibet (Theorem 10), we have that $\{\forall v\neg\theta n(x|v),\theta n\}\vdash\phi$ and $\{\forall v\neg\theta n(x|v),\theta n\}\vdash\neg\phi$. So with (2), we have that $\Gamma n,\forall v\neg\theta n(x|v)\vdash\phi$ and $\Gamma n,\forall v\neg\theta n(x|v)\vdash\neg\phi$, by (∃E). By Cut (Theorem 11), $\Gamma n\vdash\phi$ and $\Gamma n\vdash\neg\phi$. So Γn is inconsistent, contradicting the assumption. So Γ′ is consistent.

Applying the Lindenbaum Lemma (Theorem 13), let Γ″ be a maximally consistent set of sentences (of the expanded language) that contains Γ′. So, of course, Γ″ contains Γ. We can now define an interpretation M such that M satisfies every member of Γ″.

If we did not have a sign for identity in the language, we would let the domain of M be the collection of new constants $\{c0, c1, \ldots\}$. But as it is, there may be a sentence in the form $ci=cj$, with $i \neq j$, in Γ''. If so, we cannot have both ci and cj in the domain of the interpretation (as they are distinct constants). So we define the domain d of M to be the set $\{ci \mid$ there is no $j<i$ such that $ci=cj$ is in $\Gamma''\}$. In other words, a constant ci is in the domain of M if Γ'' does not declare it to be identical to an earlier constant in the list. Notice that for each new constant ci, there is exactly one $j \leq i$ such that cj is in d and the sentence $ci=cj$ is in Γ''.

We now define the interpretation function I. Let a be any constant in the expanded language. By (=I) and (∃I), $\Gamma'' \vdash \exists x x=a$, and so $\exists x x=a \in \Gamma''$. By the construction of Γ', there is a sentence in the form $(\exists x x=a \rightarrow ci=a)$ in Γ''. We have that $ci=a$ is in Γ''. As above, there is exactly one cj in d such that $ci=cj$ is in Γ''. Let $I(a)=cj$. Notice that if ci is a constant in the domain d, then $I(ci)=ci$. That is each ci in d denotes itself.

Let P be a zero-place predicate letter in K. Then I(P) is truth if P is in Γ'' and I(P) is falsehood otherwise. Let Q be a one-place predicate letter in K. Then I(Q) is the set of constants $\{ci \mid ci$ is in d and the sentence Qc is in $\Gamma''\}$. Let R be a binary predicate letter in K. Then I(R) is the set of pairs of constants $\{\langle ci,cj\rangle \mid ci$ is in d, cj is in d, and the sentence Rcicj is in $\Gamma''\}$. Three-place predicates, etc. are interpreted similarly. In effect, I interpret the non-logical terminology as they are in Γ''.

The variable assignments are similar. If v is a variable, then $s(v)=ci$, where ci is the first constant in d such that $ci=v$ is in Γ''.
The final item in this proof is a lemma that for every formula θ in the expanded language, $M \vDash \theta$ if and only if θ is in Γ''. This proceeds by induction on the complexity of θ. The case where θ is atomic follows from the definitions of M (i.e., the domain d and the interpretation function I, and the variable assignment s). The other cases follow from the various clauses in the definition of satisfaction.

Since $\Gamma \subseteq \Gamma''$, we have that M satisfies every member of Γ. By Theorem 15, the restriction of M to the original language L1K= and s also satisfies every member of Γ. Thus Γ is satisfiable.

A converse to Soundness (Theorem 18) is a straightforward corollary:

Theorem 21. For any sentence θ and set Γ of sentences, if $\Gamma \vDash \theta$, then $\Gamma \vdash D\theta$.

Proof: Suppose that $\Gamma \vDash \theta$. Then there is no interpretation M such that M satisfies every member of Γ but does not satisfy θ. So the set $\Gamma, \neg\theta$ is not satisfiable. By Completeness (Theorem 20), $\Gamma, \neg\theta$ is inconsistent. So there is a sentence ϕ such that $\Gamma, \neg\theta \vdash \phi$ and $\Gamma, \neg\theta \vdash \neg\phi$. By (¬I), $\Gamma \vdash \neg\neg\theta$, and by (DNE) $\Gamma \vdash \theta$.

Our next item is a corollary of Theorem 9, Soundness (Theorem 18), and Completeness:

Corollary 22. Compactness. A set Γ of sentences is satisfiable if and only if every finite subset of Γ is satisfiable.

Proof: If M satisfies every member of Γ, then M satisfies every member of each finite subset of Γ. For the converse, suppose that Γ is not satisfiable. Then we show that some finite subset of Γ is not satisfiable. By Completeness (Theorem 20), Γ is inconsistent. By Theorem 9 (and Weakening), there is a finite subset $\Gamma' \subseteq \Gamma$ such that Γ' is inconsistent. By Corollary 19,Γ' is not satisfiable.

Soundness and completeness together entail that an argument is deducible if and only if it is valid, and a set of sentences is consistent if and only if it is satisfiable. So we can go back and forth between model-theoretic and proof-theoretic notions, transferring properties of one to the other. Compactness holds in the model theory because all derivations use only a finite number of premises.

Recall that in the proof of Completeness (Theorem 20), we made the simplifying assumption that the set K of non-logical constants is either finite or denumerably infinite. The interpretation we produced was itself either finite or denumerably infinite. Thus, we have the following:

Corollary 23. Löwenheim-Skolem Theorem. Let Γ be a satisfiable set of sentences of the language L1K=. If Γ is either finite or denumerably infinite, then Γ has a model whose domain is either finite or denumerably infinite.

In general, let Γ be a satisfiable set of sentences of L1K=, and let κ be the larger of the size of Γ and denumerably infinite. Then Γ has a model whose domain is at most size κ.

There is a stronger version of Corollary 23. Let M1=⟨d1,I1⟩ and M2=⟨d2,I2⟩ be interpretations of the language L1K=. Define M1 to be a submodel of M2 if d1⊆d2,I1(c)=I2(c) for each constant c, and I1 is the restriction of I2 to d1. For example, if R is a binary relation letter in K, then for all a,b in d1, the pair ⟨a,b⟩ is in I1(R) if and only if ⟨a,b⟩ is in I2(R). If we had included function letters among the non-logical terminology, we would also require that d1 be closed under their interpretations in M2. Notice that if M1 is a submodel of M2, then any variable-assignment on M1 is also a variable-assignment on M2.

Say that two interpretations M1=⟨d1,I1⟩,M2=⟨d2,I2⟩ are equivalent if one of them is a submodel of the other, and for any formula of the language and any variable-assignment s on the submodel, M1,s⊨θ if and only if M2,s⊨θ. Notice that if two interpretations are equivalent, then they satisfy the same sentences.

Theorem 25. Downward Löwenheim-Skolem Theorem. Let M=⟨d,I⟩ be an interpretation of the language L1K=. Let d1 be any subset of d, and let κ be the maximum of the size of K, the size of d1, and denumerably infinite. Then there is a submodel M′=⟨d′,I′⟩ of M such that (1) d′ is not larger than κ, and (2) M and M′ are equivalent. In particular, if the set K of non-logical terminology is either finite or denumerably infinite, then any interpretation has an equivalent submodel whose domain is either finite or denumerably infinite.

Proof: Like completeness, this proof is complex, and we rest content with a sketch. The downward Löwenheim-Skolem theorem invokes the axiom of choice, and indeed, is equivalent to the axiom of choice (see the entry on the axiom of choice). So let C be a choice function on the powerset of d, so that for each non-empty subset $e \subseteq d$, $C(e)$ is a member of e. We stipulate that if e is the empty set, then $C(e)$ is $C(d)$.

Let s be a variable-assignment on M, let θ be a formula of L1K=, and let v be a variable. Define the v-witness of θ over s, written $w_v(\theta,s)$, as follows: Let q be the set of all elements $c \in d$ such that there is a variable-assignment s' on M that agrees with s on every variable except possibly v, such that $M,s' \vDash \theta$, and $s'(v)=c$. Then $w_v(\theta,s)=C(q)$. Notice that if $M,s \vDash \exists v\theta$, then q is the set of elements of the domain that can go for v in θ. Indeed, $M,s \vDash \exists v\theta$ if and only if q is non-empty. So if $M,s \vDash \exists v\theta$, then $w_v(\theta,s)$ (i.e., $C(q)$) is a chosen element of the domain that can go for v in θ. In a sense, it is a "witness" that verifies $M,s \vDash \exists v\theta$.

If e is a non-empty subset of the domain d, then define a variable-assignment s to be an e-assignment if for all variables u, $s(u)$ is in e. That is, s is an e-assignment if s assigns an element of e to each variable. Define sk(e), the Skolem-hull of e, to be the set:

$e \cup \{w_v(\theta,s) \mid \theta$ is a formula in L1K=, v is a variable, and s is an e-assignment$\}$.

That is, the Skolem-Hull of e is the set e together with every v-witness of every formula over every e-assignment. Roughly, the idea is to start with e and then throw in enough elements to make each existentially quantified formula true. But we cannot rest content with the Skolem-hull, however. Once we throw the "witnesses" into the domain, we need to deal with sk(e) assignments. In effect, we need a set which is its own Skolem-hull, and also contains the given subset d_1.

We define a sequence of non-empty sets $e_0, e_1, \dots$ as follows: if the given subset d_1 of d is empty and there are no constants in K, then let e_0 be $C(d)$, the choice function applied to the entire domain; otherwise let e_0 be the union of d_1 and the denotations under I of the constants in K. For each natural number n, e_{n+1} is $sk(e_n)$. Finally, let d' be the union of the sets e_n, and let I' be the restriction of I to d'. Our interpretation is $M'=\langle d',I' \rangle$.

Clearly, d1 is a subset of d′, and so M′ is a submodel of M. Let κ be the maximum of the size of K, the size of d1, and denumerably infinite. A calculation reveals that the size of d′ is at most κ, based on the fact that there are at most κ-many formulas, and thus, at most κ-many witnesses at each stage. Notice, incidentally, that this calculation relies on the fact that a denumerable union of sets of size at most κ is itself at most κ. This also relies on the axiom of choice.

The final item is to show that M′ is equivalent to M: For every formula θ and every variable-assignment s on M′,

M,s⊨θ if and only if M′,s⊨θ.

The proof proceeds by induction on the complexity of θ. Unfortunately, space constraints require that we leave this step as an exercise.

Another corollary to Compactness (Corollary 22) is the opposite of the Löwenheim-Skolem theorem:

Theorem 26. Upward Löwenheim-Skolem Theorem. Let Γ be any set of sentences of L1K=, such that for each natural number n, there is an interpretation Mn=⟨dn,In⟩, and an assignment sn on Mn, such that dn has at least n elements, and Mn,sn satisfies every member of Γ. In other words, Γ is satisfiable and there is no finite upper bound to the size of the interpretations that satisfy every member of Γ. Then for any infinite cardinal κ, there is an interpretation M=⟨d,I⟩ and assignment s on M, such that the size of d is at least κ and M,s satisfies every member of Γ. In particular, if Γ is a set of sentences, then it has arbitrarily large models.

Proof: Add a collection of new constants {cα|α<κ}, of size κ, to the language, so that if c is a constant in K, then cα is different from c, and if α<β<κ, then cα is a different constant than cβ. Consider the set of formulas Γ′ consisting of Γ together with the set {¬cα=cβ|α≠β}. That is, Γ′ consists of Γ together with statements to the effect that any two different new constants denote different objects. Let Γ″ be any finite subset of Γ′, and let m be the number of new constants that occur in Γ″. Then expand the interpretation Mm to an interpretation M′m of the new language, by interpreting each of the new constants in Γ″ as a different member of the domain dm. By hypothesis, there are enough members of dm to do this. One can interpret the other new constants at will. So Mm is a restriction of M′m. By hypothesis (and Theorem 15), M′m,sm satisfies every member of Γ. Also M′m,sm satisfies the members of {¬cα=cβ|α≠β} that are in Γ″. So M′m,sm satisfies every member of Γ″. By compactness, there is an interpretation M=⟨d,I⟩ and an assignment s on M such that M,s satisfies every member of Γ′. Since Γ′ contains every member of {¬cα=cβ|α≠β}, the domain d of M must be of size at least κ, since each of the new constants must have a different denotation. By Theorem 15, the restriction of M to the original language L1K= satisfies every member of Γ, with the variable-assignment s.

Combined, the proofs of the downward and upward Löwenheim-Skolem theorems show that for any satisfiable set Γ of sentences, if there is no finite bound on the models of Γ, then for any infinite cardinal κ, there is a model of Γ whose domain has size exactly κ. Moreover, if M is any interpretation whose domain is infinite, then for any infinite cardinal κ, there is an interpretation M′ whose domain has size exactly κ such that M and M′ are equivalent.

These results indicate a weakness in the expressive resources of first-order languages like L1K=. No satisfiable set of sentences can guarantee that its models are all denumerably infinite, nor can any satisfiable set of sentences guarantee that its models are uncountable. So in a sense, first-order languages cannot express the notion of “denumerably infinite”, at least not in the model theory. (See the entry on second-order and higher-order logic.)

Let A be any set of sentences in a first-order language L1K=, where K includes terminology for arithmetic, and assume that every member of A is true of the natural numbers. We can even let A be the set of all sentences in L1K= that are true of the natural numbers. Then A has uncountable models, indeed models of any infinite cardinality. Such interpretations are among those that are sometimes called unintended, or non-standard models of arithmetic. Let B be any set of first-order sentences that are true of the real numbers, and let C be any first-order axiomatization of set theory. Then if B and C are satisfiable (in infinite interpretations), then each of them has denumerably infinite models. That is, any first-order, satisfiable set theory or theory of the real numbers, has (unintended) models the size of the natural numbers. This is despite the fact that a sentence (seemingly) stating that the universe is uncountable is provable in most set-theories. This situation, known as the Skolem paradox, has generated much discussion, but we must refer the reader elsewhere for a sample of it (see the entry on Skolem's paradox and Shapiro 1996).

12.6 THE ONE RIGHT LOGIC?

Logic and reasoning go hand in hand. We say that someone has reasoned poorly about something if they have not reasoned logically, or that an argument is bad because it is not logically valid. To date, research has been devoted to exactly just what types of logical systems are appropriate for guiding our reasoning. Traditionally, classical logic has been the logic suggested as the ideal for guiding reasoning (for example, see Quine [1986], Resnik [1996] or Rumfitt [2015]). For this reason, classical logic has often been called "the one right logic". See Priest [2006a] for a description of how being the best reasoning-guiding logic could make a logic the one right logic.

That classical logic has been given as the answer to which logic ought to guide reasoning is not unexpected. It has rules which are more or less intuitive, and is surprisingly simple for how strong it is. Plus, it is both sound and complete, which is an added bonus. There are some issues, though. As indicated in Section 5, there are certain expressive limitations to classical logic. Thus, much literature has been written challenging this status quo. This literature in general stems from three positions. The first is that classical logic is not reason-guiding because some other single logic is. Examples of this type of argument can be found in Brouwer [1949], Heyting [1956] and Dummett [2000] who argue that intuitionistic logic is correct, and Anderson and Belnap [1975], who argue relevance logic is correct, among many others. Further, some people propose that an extension of classical logic which can express the notion of "denumerably infinite" (see Shapiro [1991]). The second objection to the claim that classical logic is the one right logic comes from a different perspective: logical pluralists claim that classical logic is not the (single) one right logic, because more than one logic is right. See Beall and Restall [2006] and Shapiro [2014] for examples of this type of view (see also the entry on logical pluralism). Finally, the last objection to the claim that classical logic is the one right logic is that logic(s) is not reasoning-guiding, and so there is no one right logic.

Suffice it to say that, though classical logic has traditionally been thought of as "the one right logic", this is not accepted by everyone. An interesting feature of these debates, though, is that they demonstrate clearly the strengths and weaknesses of various logics (including classical logic) when it comes to capturing reasoning.

Check Your Progress 2

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. What do you know the Meta-theory?

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2. Discuss the One Right Logic.

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12.7 LET US SUM UP

In Symbolic Logic (1932), C. I. Lewis developed five modal systems S1 − S5. S4 and S5 are so-called normal modal systems. Since Lewis and

Langford's pioneering work many other systems of this kind have been investigated, among them the 32 systems that can be generated by the five axioms T, D, B, 4 and 5. Lewis also discusses how his systems can be augmented by propositional quantifiers and how these augmented logics allow us to express some interesting ideas that cannot be expressed in the corresponding quantifier-free logics. In this paper, I will develop 64 normal modal semantic tableau systems that can be extended by propositional quantifiers yielding 64 extended systems. All in all, we will investigate 128 different systems. I will show how these systems can be used to prove some interesting theorems and I will discuss Lewis's so-called existence postulate and some of its consequences. Finally, I will prove that all normal modal systems are sound and complete and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete.

12.8 KEY WORDS

Symbolic Logic: the use of symbols to denote propositions, terms, and relations in order to assist reasoning.

12.9 QUESTIONS FOR REVIEW

1. Discuss the Building blocks.
2. Describe Atomic formulas.
3. Discuss the Compound formulas.
4. Write about Features of the syntax.

12.10 SUGGESTED READINGS AND REFERENCES

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12.11 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. See Section 12.2
2. See Section 12.3
3. See Section 12.4

Check Your Progress 2

1. See Section 12.5
2. See Section 12.6

UNIT 13: MODERN ORIGINS OF MODAL LOGIC: THE SYSTEMS OF T.S4 AND S5

STRUCTURE

13.0 OBJECTIVES

After this unit, we can able to know:

- The Syntactic Tradition
- The Matrix Method and Some Algebraic Results
- The Model Theoretic Tradition

13.1 INTRODUCTION

Modal logic can be viewed broadly as the logic of different sorts of modalities, or modes of truth: alethic ("necessarily"), epistemic ("it is known that"), deontic ("it ought to be the case that"), or temporal ("it has been the case that") among others. Common logical features of these operators justify the common label. In the strict sense however, the term "modal logic" is reserved for the logic of the alethic modalities, as opposed for example to temporal or deontic logic. From a merely technical point of view, any logic with non-truth-functional operators, including first-order logic, can be regarded as a modal logic: in this perspective the quantifiers too can be regarded as modal operators (as in Montague 1960). Nonetheless, we follow the traditional understanding of modal logics as not including full-fledged first-order logic. In this perspective it is the modal operators that can be regarded as restricted quantifiers, ranging over special entities like possible worlds or temporal instants. Arthur Prior was one of the first philosophers/logicians to emphasize that the modal system S5 can be translated into a fragment of first-order logic, which he called "the uniform monadic first-order predicate calculus" (Prior and Fine 1977: 56). Monadic, since no relations between worlds needs to be stated for S5; and uniform as only one variable is needed to quantify over worlds (instants) when bound, and to refer to the privileged state (the actual world or the present time) when free (see Prior and Fine 1977). Concerning the technical question of which model-theoretic features characterize modal logics understood as well-behaved fragments of first-order logic, see Blackburn and van Benthem's "Modal Logic: A Semantic Perspective" (2007a).

The scope of this entry is the recent historical development of modal logic, strictly understood as the logic of necessity and possibility, and particularly the historical development of systems of modal logic, both syntactically and semantically, from C.I. Lewis's pioneering work starting in 1912, with the first systems devised in 1918, to S. Kripke's work in the early 1960s. In that short span of time of less than fifty years, modal logic flourished both philosophically and mathematically. Mathematically, different modal systems were developed and advances in algebra helped to foster the model theory for such systems. This culminated in the development of a formal semantics that extended to modal logic the successful first-order model theoretic techniques, thereby affording completeness and decidability results for many, but not all, systems. Philosophically, the availability of different systems and the adoption of the possible worlds model-theoretic semantics were naturally accompanied by reflections on the nature of possibility and necessity, on distinct sorts of necessities, on the role of the formal semantics, and on the nature of the possible worlds, to mention just a few. In particular, the availability of different systems brings to the fore the philosophical question of which modal logic is the correct one, under some intended interpretation of the modal operators, e.g., as logical or metaphysical necessity. Questions concerning the interpretability of modal logic, especially quantified modal logic, were insistently raised by Quine. All such questions are not pursued in this entry which is mostly devoted to the formal development of the subject.

Modal logic is a rich and complex subject matter. This entry does not present a complete survey of all the systems developed and of all the model theoretic results proved in the lapse of time under consideration. It does however offer a meaningful survey of the main systems and aims to be useful to those looking for an historical outline of the subject matter that, even if not all-inclusive, delineates the most interesting model theoretic results and indicates further lines of exploration. Bull and Segerberg's (1984: 3) useful division of the original sources of modal logic into three distinct traditions—syntactic, algebraic, and model theoretic—is adopted. For other less influential traditions see Bull and Segerberg (1984: 16). See also Lindström and Segerberg's "Modal Logic and Philosophy" (2007). The main focus of this entry is on propositional modal logic, while only some particular aspects of the semantics of quantified modal logic are discussed. For a more detailed treatment of quantified modal logic, consult the SEP entry on modal logic. Concerning the entry's notation, notice that ⇒ is adopted in place of Lewis's fishhook for strict implication, and ⇔ for strict equivalence.

Modal logic deals with modal concepts, such as necessity, possibility and contingency, and with the logical relationships between propositions that include such concepts. Modal logicians study various modal principles, arguments and systems (Blackburn et. al. 2001; 2007, Chellas 1980, Fitting and Mendelsohn 1998, Garson 2006, Hughes and Cresswell 1968; 1996, Kracht 1999, Priest 2008). Lewis and Langford's Symbolic Logic (1932) marks the beginning of modern, symbolic, modal logic. The purpose of this paper is to develop 64 so-called normal modal semantic tableau systems (half of them correspond to the 32 axiomatic systems that can be generated by the well-known axioms T, D, B, 4 and 5) and to show how these systems can be augmented by propositional quantifiers.

Therefore, we will consider 128 different systems in this paper. The tableau rules T − T, T − D, T − B, T − 4 and T − 5 and the 32 systems that can be generated from these rules are well-known. The normal modal tableau system that includes T − T and T − 4 is deductively equivalent with, that is, includes the same theorems as, Lewis's system S4, and the normal modal tableau system that includes T − T, T − B and T − 4 is deductively equivalent with Lewis's system S5. 1 All systems in this paper are stronger than Lewis's systems S1 − S3. The tableau rule T − F, which is especially interesting for our purposes, is much less well-known. Hence, all systems that contain this rule deserve extra attention. The propositional part of all systems in this paper is fairly standard, but as far as I know there are no modal tableau systems in the literature that include propositional quantifiers of the kind that is used in our formal language.2 Hence, all 64 extended systems are new. Furthermore, I will show how these systems can be used to prove some interesting theorems that contain propositional quantifiers and I will discuss Lewis's so-called existence postulate and some of its consequences. According to this postulate, there is some pair of propositions X and Y , so related that X implies nothing about the truth or falsity of Y (Lewis and Langford 1932, 179). This postulate can be symbolised in the following way: $\exists X \exists Y (\neg \Box(X \rightarrow Y) \wedge \neg \Box(X \rightarrow \neg Y))$. Finally, I will prove that all normal modal systems are sound and complete and that all systems (including the extended systems) are sound with respect to their semantics. It is left as an open question whether or not the extended systems are complete. Since all extended systems in this paper are new, there are good logical reasons to be interested in our results. There are also several philosophically interesting reasons. In systems with propositional quantifiers we can express many ideas that cannot be expressed in any quantifier-free normal modal systems. We can, for example, symbolise Lewis's existence postulate, from which it follows that there is something that is contingent, that material implication does not coincide with necessary implication, and that there are at least four distinct propositions, among other things. In ordinary normal modal systems, we cannot prove any of these propositions; we cannot even find plausible formalisations of them. Furthermore, the tableau systems are often more user-friendly than their axiomatic counterparts, it is often easier to prove something in a tableau system than in an axiomatic system and it is often easier to derive a sentence from a set of premises. Consequently, there are both good philosophical and technical reasons to be interested in the systems in this paper. (For more information on propositional quantifiers, see, for example (Lewis and Langford 1932, 178−198), (Kripke 1959), (Bull 1969), (Fine 1970), (Kaplan 1970), (Gabbay 1971) and (Gallin 1975).)

13.2 THE SYNTACTIC TRADITION

In a 1912 pioneering article in Mind "Implication and the Algebra of Logic" C.I. Lewis started to voice his concerns on the so-called "paradoxes of material implication". Lewis points out that in Russell and Whitehead's Principia Mathematica we find two "startling theorems: (1) a false proposition implies any proposition, and (2) a true proposition is implied by any proposition" (1912: 522). In symbols:

$\neg p \to (p \to q)$(1)

and

$p \to (q \to p)$(2)

Lewis has no objection to these theorems in and of themselves:

In themselves, they are neither mysterious sayings, nor great discoveries, nor gross absurdities. They exhibit only, in sharp outline, the meaning of "implies" which has been incorporated into the algebra. (1912: 522)

However, the theorems are inadequate vis-à-vis the intended meaning of "implication" and our actual modes of inference that the intended meaning tries to capture. So Lewis has in mind an intended meaning for the conditional connective → or ⊃, and that is the meaning of the English word "implies". The meaning of "implies" is that "of ordinary inference and proof" (1912: 531) according to which a proposition implies another proposition if the second can be logically deduced from the first. Given such an interpretation, (1) and (2) ought not to be theorems, and propositional logic may be regarded as unsound vis-à-vis the reading of → as logical implication. Consider (2) for example: from the sheer truth of a proposition p it does not (logically) follow that p follows logically from any proposition whatsoever. Additionally, given the intended, strict reading of → as logical implication and the equivalence of (¬p→q) and (p∨q), Lewis infers that disjunction too must be given a new intensional sense, according to which (p∨q) holds just in case if p were not the case it would have to be the case that q.

Considerations of this sort, based on the distinction between extensional and intensional readings of the connectives, were not original to Lewis. Already in 1880 Hugh MacColl in the first of a series of eight papers on Symbolical Reasoning published in Mind claimed that (p→q) and (¬p∨q) are not equivalent: (¬p∨q) follows from (p→q), but not vice versa (MacColl 1880: 54). This is the case because MacColl interprets ∨ as regular extensional disjunction, and → as intensional implication, but then from the falsity of p or the truth of q it does not follow that p without q is logically impossible. In the second paper of the series, MacColl distinguishes between certainties, possibilities and variable statements, and introduces Greek letters as indices to classify propositions. So αε expresses that α is a certainty, αη that α is an impossibility, and αθ that α is a variable, i.e., neither a certainty nor an impossibility (MacColl 1897: 496–7). Using this threefold classification of statements, MacColl proceeds to distinguish between causal and general implication. A causal implication holds between statements α and β if whenever α is true β is true, and β is not a certainty. A general implication holds between α and β whenever α and not−β is impossible, thus in particular whenever α is an impossibility or β a certainty (1897: 498). The use of indices opened the door to the iteration of modalities, and the beginning of the third paper of the series (MacColl 1900: 75–6) is devoted to clarify the meaning of statements with iterated indices, including τ for truth and ι for negation. So for example Aηιε is read as "It is certain that it is false that A is impossible" (note that the indices are read from right to left). Interestingly, Bertrand Russell's 1906 review of MacColl's book Symbolic Logic and its Applications (1906) reveals that Russell did not understand the modal idea of the variability of a proposition, hence wrongly attributed to MacColl a confusion between sentences and propositions which allowed the attribution of variability only to sentences whose meaning, hence truth value, was not fixed. Similarly, certainty and impossibility are for Russell material properties of propositional functions (true of everything or of nothing) and not modal properties of propositions. It might be said that MacColl's work came too early and fell on deaf ears. In fact, Rescher reports on Russell's declared difficulty in understanding MacColl's symbolism and, more importantly, argues that Russell's view of logic had a negative impact on the development of modal logic ("Bertrand Russell and Modal Logic" in Rescher 1974: 85–96). Despite MacColl's earlier work, Lewis can be regarded as the father of the syntactic tradition, not only because of his influence on later logicians, but especially because of his introduction of various systems containing the new intensional connectives.

13.2.1 The Lewis Systems

In "The Calculus of Strict Implication" (1914) Lewis suggests two possible alternatives to the extensional system of Whitehead and Russell's Principia Mathematica. One way of introducing a system of strict implication consists in eliminating from the system those theorems that, like (1) and (2) above, are true only for material implication but not for strict implication, thereby obtaining a sound system for both material and strict implication, but in neither case complete. The second, more fruitful alternative consists in introducing a new system of strict implication, still modeled on the Whitehead and Russell system of material implication, that will contain (all or a part of) extensional propositional logic as a proper part, but aspiring to completeness for at least strict implication. This second option is further developed in A Survey of Symbolic Logic (1918). There Lewis introduces a first system meant to capture the ordinary, strict sense of implication, guided by the idea that:

Unless "implies" has some "proper" meaning, there is no criterion of validity, no possibility even of arguing the question whether there is one or not. And yet the question What is the "proper" meaning of "implies"? remains peculiarly difficult. (1918: 325)

The 1918 system takes as primitive the notion of impossibility (¬&)(¬&), defines the operator of strict implication in its terms, and still employs an operator of intensional disjunction. However, Post will prove that this system leads to the collapse of necessity to truth—alternatively, of impossibility to falsity—since from one of its theorems ((p⇒q)⇔(¬&q⇒¬&p))((p⇒q)⇔(¬&q⇒¬&p)) it can be proved that (¬p⇔¬&p)(¬p⇔¬&p). In 1920, "Strict Implication—An Emendation", Lewis fixes the system substituting for the old axiom the weaker one: ((p⇒q)⇒(¬&q⇒¬&p))((p⇒q)⇒(¬&q⇒¬&p)). Finally, in Appendix II of the Lewis and Langford's volume Symbolic Logic (1932: 492–502) "The Structure of the System of Strict Implication" the 1918 system is given a new axiomatic base.

In the 1932 Appendix C.I. Lewis introduces five different systems. The modal primitive symbol is now the operator of possibility &&, strict implication (p⇒q)(p⇒q) is defined as ¬&(p∧¬q)¬&(p∧¬q), and ∨∨ is ordinary extensional disjunction. The necessity operator ϒ⌋ can also be introduced and defined, though Lewis does not, in the usual way as ¬&←←&¬.

Where p,qp,q, and rr are propositional variables, System **S1** has the following axioms:

Axioms for **S1**

(p∧q)(p∧q)p((p∧q)∧r)p((p⇒q)∧(q⇒r))(p∧(p⇒q))⇒(q∧p)⇒p⇒(p∧p)⇒(p∧(q∧r))⇒¬¬p⇒(p⇒r)⇒q(B1)(B2)(B3)(B4)(B5)(B6)(B7)(B1)(p∧q)⇒(q∧p)(B2)(p∧q)⇒p(B3)p⇒(p∧p)(B4)((p∧q)∧r)⇒(p∧(q∧r))(B5)p⇒¬¬p(B6)((p⇒q)∧(q⇒r))⇒(p⇒r)(B7)(p∧(p⇒q))⇒q

Axiom B5 was proved redundant by McKinsey (1934), and can thereby be ignored.

The rules are (1932: 125–6):

Rules for **S1**

Uniform Substitution

A valid formula remains valid if a formula is uniformly substituted in it for a propositional variable.

Substitution of Strict Equivalents

Either of two strictly equivalent formulas can be substituted for one another.

Adjunction

If ΦΦ and ΨΨ have been inferred, then Φ∧ΨΦ∧Ψ may be inferred.

Strict Inference

If ΦΦ and Φ⇒ΨΦ⇒Ψ have been inferred, then ΨΨ may be inferred.

System **S2** is obtained from System **S1** by adding what Lewis calls “the consistency postulate”, since it obviously holds for && interpreted as consistency:

&(p∧q)⇒&p(B8)(B8)&(p∧q)⇒&p

System **S3** is obtained from system **S1** by adding the axiom:

((p⇒q)⇒(¬&q⇒¬&p))(A8)(A8)((p⇒q)⇒(¬&q⇒¬&p))

System **S3** corresponds to the 1918 system of A Survey, which Lewis originally considered the correct system for strict implication. By 1932, Lewis has come to prefer system **S2**. The reason, as reported in Lewis 1932: 496, is that both Wajsberg and Parry derived in system **S3**—in its 1918 axiomatization—the following theorem:

(p⇒q)⇒((q⇒r)⇒(p⇒r)),(p⇒q)⇒((q⇒r)⇒(p⇒r)),

which according to Lewis ought not to be regarded as a valid principle of deduction. In 1932 Lewis is not sure that the questionable theorem is not derivable in **S2**. Should it be, he would then adjudicate **S1** as the proper system for strict implication. However, Parry (1934) will later prove that neither A8 nor

(p⇒q)⇒((q⇒r)⇒(p⇒r))(p⇒q)⇒((q⇒r)⇒(p⇒r))

can be derived in **S2**.

A further existence axiom can be added to all these systems:

(∃p,q)(¬(p⇒q)∧¬(p⇒¬q))(B9)(B9)(∃p,q)(¬(p⇒q)∧¬(p⇒¬q))

The addition of B9 makes it impossible to interpret ⇒⇒ as material implication, since in the case of material implication it can be proved that for any

propositions pp and q,((p→q)∨(p→¬q))q,((p→q)∨(p→¬q)) (1932: 179).

From B9 Lewis proceeds to deduce the existence of at least four logically distinct propositions: one true and necessary, one true but not necessary, one false and impossible, one false but not impossible (1932: 184–9).

Following Becker (1930), Lewis considers three more axioms:

Three additional axioms

¬&¬p&πp⇒¬&←←&¬p⇒←&←&p⇒¬&←&p(C10)(C11)(C12)(C10)

←&¬p⇒¬&←←&¬p(C11)&p⇒¬&←&p(C12)p⇒¬&←&p

System **S4** adds axiom C10 to the basis of **S1**. System **S5** adds axiom C11, or alternatively C10 and C12, to the basis of **S1**. Lewis concludes Appendix II by noting that the study of logic is best served by focusing on systems weaker than **S5** and not exclusively on **S5**.

Paradoxes of strict implication similar to those of material implication arise too. Given that strict implication $(p \Rightarrow q)$ is defined as $\neg \& (p \wedge \neg q)$, it follows that an impossible proposition implies anything, and that a necessary proposition is implied by anything. Lewis argues that this is as it ought to be. Since impossibility is taken to be logical impossibility, i.e., ultimately a contradiction, Lewis argues that from an impossible proposition like $(p \wedge \neg p)$, both p and $\neg p$ follow. From p we can derive $(p \vee q)$, for any proposition q. From $\neg p$ and $(p \vee q)$, we can derive q (1932: 250). The argument is controversial since one might think that the principle $(p \Rightarrow (p \vee q))$ should not be a theorem of a system aiming to express ordinary implication (see, e.g., Nelson 1930: 447). Whatever the merits of this argument, those who disagreed with Lewis started to develop a logic of entailment based on the assumption that entailment requires more than Lewis's strict implication. See, for example, Nelson 1930, Strawson 1948, and Bennett 1954. See also the SEP entry on relevance logic.

Notice that it was Lewis's search for a system apt to express strict implication that made Quine reject modal systems as based on a use-mention confusion insofar as such systems were formulated to express at the object level proof-theoretic or semantic notions like consistency, implication, derivability and theoremhood (in fact, whenever $p \rightarrow q$ is a propositional theorem, system **S1**, and so all the other stronger Lewis systems too, can prove $p \Rightarrow q$ (Parry 1939: 143)).

13.2.2 Other Systems and Alternative Axiomatizations of the Lewis Systems

Gödel in "An Interpretation of the Intuitionistic Propositional Calculus" (1933) is the first to propose an alternative axiomatization of the Lewis system S4 that separates the propositional basis of the system from the modal axioms and rules. Gödel adds the following rules and axioms to the propositional calculus.

If ⊢α⊢ϒ(p→q)⊢ϒp⊢ϒp then ⊢ϒα,→(ϒp→ϒq),→p, and→□□p.(Necessitation)(Axiom K)(Axiom T)(Axiom 4)

Initially, Gödel employs an operator B of provability to translate Heyting's primitive intuitionistic connectives, and then observes that if we replace B with an operator of necessity we obtain the system S4. Gödel also claims that a formula □p∨□q is not provable in S4 unless either □p or □q is provable, analogously to intuitionistic disjunction. Gödel's claim will be proved algebraically by McKinsey and Tarski (1948). Gödel's short note is important for starting the fruitful practice of axiomatizing modal systems separating the propositional calculus from the strictly modal part, but also for connecting intuitionistic and modal logic.

Feys (1937) is the first to propose system T by subtracting axiom 4 from Gödel's system S4 (see also Feys 1965: 123–124). In An Essay in Modal Logic (1951) von Wright discusses alethic, epistemic, and deontic modalities, and introduces system M, which Sobociński (1953) will prove to be equivalent to Feys' system T. Von Wright (1951: 84–90) proves that system M contains Lewis's S2, which contains S1—where system S is said to contain system S′ if all the formulas provable in S′ can be proved in S too. System S3, an extension of S2, is not contained in M. Nor is M contained in S3. Von Wright finds S3 of little independent interest, and sees no reason to adopt S3 instead of the stronger S4. In general, the Lewis systems are numbered in order of strength, with S1 the weakest and S5 the strongest, weaker systems being contained in the stronger ones.

Lemmon (1957) also follows Gödel in axiomatizing modal systems on a propositional calculus base, and presents an alternative axiomatization of the Lewis systems. Where PC is the propositional calculus base, PC may be characterized as the following three rules (1957: 177):

A characterization of propositional calculus PC

PCa If α is a tautology, then ⊢α

PCb Substitution for propositional variables

PCc Material detachment/Modus Ponens: if α and α→β are tautologies, then so is β

Further rules in Lemmon's system are:

(a) If ⊢α then ⊢□α(Necessitation)

(a′) If α is a tautology or an axiom, then ⊢□α

(b) If ⊢□(α→β) then ⊢□(□α→□β)

(b′) Substitutability of strict equivalents.

Further axioms in Lemmon's system are:

ϒ(p→q)ϒ(p→q)ϒp(ϒ(p→q)∧ϒ(q→r))→ϒ(ϒp→ϒq)→(ϒp→ϒq)→p→ϒ(p→r)(Axiom K)(Axiom T)(1)(1')(2)(3)

Using the above rules and axioms Lemmon defines four systems. System P1, which is proved equivalent to the Lewis system S1, employs the propositional basis (PC), rules (a′)—necessitation of tautologies and axioms—and (b′), and axioms (2) and (3). System P2, equivalent to S2, employs (PC), rules (a′) and (b), and axioms (2) and (1′). System P3, equivalent to S3, employs (PC), rule (a′), and axioms (2) and (1). System P4, equivalent to S4, employs (PC), rule (a), and axioms (2) and (1). In Lemmon's axiomatization it is easy to see that S3 and von Wright's system M (Feys' T) are not included in each other, given M's stronger rule of necessitation and S3's stronger axiom (1) in place of (1′) = K. In general, Lemmon's axiomatization makes more perspicuous the logical distinctions between the different Lewis systems.

Lemmon considers also some systems weaker than S1. Of particular interest is system S0.5 which weakens S1 by replacing rule (a′) with the weaker rule (a″):

(a″) If α is a tautology, then ⊢□α.

Lemmon interprets system S0.5 as a formalized metalogic of the propositional calculus, where □α is interpreted as "α is a tautology".

We call "normal" the systems that include PC, axiom K and the rule of necessitation. System K is the smallest normal system. System T adds axiom T to system K. System B (the Brouwersche system) adds axiom B

⊢p⇒ϒ&p(equivalent to Becker's C12)

to system T. S4 adds axiom 4 (equivalent to Becker's C10) to system T. S5 adds axioms B and 4, or alternatively axiom E

′&p⇒ϒ&p(equivalent to Becker's C11)

to system T. Lewis's systems S1, S2, and S3 are non-normal given that they do not contain the rule of Necessitation. For the relationship between these (and other) systems, and the conditions on frames that the axioms impose, consult the SEP entry on modal logic.

Only a few of the many extensions of the Lewis systems that have been discussed in the literature are mentioned here. Alban (1943) introduced system S6 by adding to S2 the axiom ′&&p. Halldén (1950) calls S7 the system that adds the axiom ⊢&&p to S3, and S8 the system that extends S3 with the addition of the axiom ⊢←&←&&p. While the addition of an axiom of universal possibility ⊢&p would be inconsistemt with all the Lewis systems, since they all contain theorems of the form ⊢□p, systems S6, S7 and S8 are consistent. Instead, the addition of either of these axioms to S4, and so also to S5, results in an inconsistent system, given that in S4 ⊢&&p⇒&p. Halldén also proved that a formula is a theorem of S3 if and only if it is a theorem of both S4 and S7 (1950: 231–232), thus S4 and S7 are two alternative extensions of S3.

13.3 THE MATRIX METHOD AND SOME ALGEBRAIC RESULTS

In “Philosophical Remarks on Many-Valued Systems of Propositional Logic” (1930. But Łukasiewicz 1920 is a preliminary Polish version of the main ideas of this paper), Łukasiewicz says:

When I recognized the incompatibility of the traditional theorems on modal propositions in 1920, I was occupied with establishing the system of the ordinary “two-valued” propositional calculus by means of the matrix method. I satisfied myself at the time that all theses of the ordinary propositional calculus could be proved on the assumption that their propositional variables could assume only two values, “0” or “the false”, and “1” or “the true”. (1970: 164)

This passage illustrates well how Łukasiewicz was thinking of logic in the early twenties. First, he was thinking in algebraic terms, rather than syntactically, concerning himself not so much with the construction of new systems, but with the evaluation of the systems relatively to sets of values. Secondly, he was introducing three-valued matrices to make logical space for the notion of propositions (eminently about future contingents) that are neither true nor false, and that receive the new indeterminate value ½. Ironically, later work employing his original matrix method will show that the hope of treating modal logic as a three-valued system cannot be realized. See also the SEP entry on many-valued logic.

A matrix for a propositional logic L is given by (i) a set K of elements, the truth-values, (ii) a non-empty subset D⊆K of designated truth-values, and (iii) operations on the set K, that is functions from n-tuples of truth-values to truth-values, that correspond to the connectives of L. A matrix satisfies a formula A under an assignment σ of elements of K to the variables of A if the value of A under σ is a member of D, that is, a designated value. A matrix satisfies a formula if it satisfies it under every assignment σ. A matrix for a modal logic M extends a matrix for a propositional logic by adding a unary function that corresponds to the connective &.

Matrices are typically used to show the independence of the axioms of a system as well as their consistency. The consistency of two formulas A and B is established by a matrix that, under an assignment σ, assigns to both formulas designated values. The independence of formula B from formula A is established by a matrix that (i) preserves the validity of the rules of the system and that (ii) under an interpretation σ assigns to A but not to B a designated value. Parry (1939) uses the matrix method to show that the number of modalities of Lewis's systems S3 and S4 is finite. A modality is a modal function of one variable that contains only the operators ¬ and &. The degree of a modality is given by the number of □ operators contained. A proper modality is of degree higher than zero. Proper modalities can be of four different forms:

¬…&p&…&p¬…&¬p&…¬p.(1)(2)(3)(4)

The improper modalities are p and ¬p (1939: 144). Parry proves that S3 has 42 distinct modalities, and that S4 has 14 distinct modalities. It was already known that system S5 has only 6 distinct modalities since it reduces all modalities to modalities of degree zero or one. Parry introduces system S4.5 by adding to S4 the following axiom:

⊢¬&¬&¬&p⇒¬&p.

The system reduces the number of modalities of S4 from 14 to 12 (or 10 proper ones). The addition of the same axiom to Lewis's system S3 results in a system with 26 distinct modalities. Moreover, if we add

⊢¬&←&&p⇒¬&←&p

to S3 we obtain a distinct system with 26 modalities also intermediate between S3 and S4. Therefore the number of modalities does not uniquely determine a system. Systems S1 and S2, as well as T and B, have an infinite number of modalities (Burgess 2009, chapter 3 on Modal Logic, discusses the additional systems S4.2 and S4.3 and explains well the reduction of modalities in different systems).

A characteristic matrix for a system L is a matrix that satisfies all and only the theorems of L. A matrix is finite if its set K of truth-values is finite. A finite characteristic matrix yields a decision procedure, where a system is decidable if every formula of the system that is not a theorem is falsified by some finite matrix (this is the finite model property). Yet Dugundji (1940) shows that none of S1–S5 has a finite characteristic matrix. Hence, none of these systems can be viewed as an n-valued logic for a finite n. Later, Scroggs (1951) will prove that every proper extension of S5 that preserves detachment for material implication and is closed under substitution has a finite characteristic matrix.

Despite their lack of a finite characteristic matrix, McKinsey (1941) shows that systems S2 and S4 are decidable. To prove these results McKinsey introduces modal matrices (K,D,−,∗,×), with −, ∗, and × corresponding to negation, possibility, and conjunction respectively. A matrix is normal if it satisfies the following conditions:

if x∈D and (x⇒y)∈D and y∈K, then y∈D,

if x∈D and y∈D, then x×y∈D,

if x∈K and y∈K and x⇔y∈D, then x=y.

These conditions correspond to Lewis's rules of strict inference, adjunction and substitution of strict equivalents. The structure of McKinsey's proof is as follows. The proof employs three steps. First, using an unpublished method of Lindenbaum explained to him by Tarski which holds for systems that have the rule of Substitution for propositional variables, McKinsey shows that there is an S2-characteristic matrix M=(K,D,−,∗,×) that does not satisfy condition (iii) and is therefore non-normal. M is a trivial matrix whose domain is the set of formulas of the system, whose designated elements are the theorems of the system, and whose operations are the connectives themselves. The trivial matrix M does not satisfy (iii) given that for some distinct formulas A and B, A⇔B is an S2-theorem. Second, McKinsey shows how to construct from M a normal, but still infinite, S2-characteristic matrix M1=(K1,D1,−1,∗1,×1), whose elements are equivalence classes of provably equivalent formulas of S2, i.e., of formulas A and B such that A⇔B is a theorem of S2, and whose operations are revised accordingly. For example, if E(A) is the set of formulas provably equivalent to A and E(A)∈K1, then −1E(A)=E(−A)=E(¬A).M1 satisfies exactly the formulas satisfied by M without violating condition (iii), hence it is a characteristic normal matrix for S2 (M1 is the Lindenbaum algebra for S2). Finally, it is shown that for every formula A that is not a theorem of S2 there is a finite and normal matrix (a sub-algebra of M1) that falsifies it. A similar proof is given for S4.

A matrix is a special kind of algebra. An algebra is a matrix without a set D of designated elements. Boolean algebras correspond to matrices for propositional logic. According to Bull and Segerberg (1984: 10) the generalization from matrices to algebras may have had the effect of encouraging the study of these structures independently of their connections to logic and modal systems. The set of designated elements D in fact facilitates a definition of validity with respect to which the theorems of a system can be evaluated. Without such a set the most obvious link to logic is severed. A second generalization to classes of algebras, rather than merely to individual algebras, was also crucial to the mathematical development of the subject matter. Tarski is the towering figure in such development.

Jónsson and Tarski (1951 and 1952) introduce the general idea of Boolean algebras with operators, i.e., extensions of Boolean algebras by addition of operators that correspond to the modal connectives. They prove a general representation theorem for Boolean algebras with operators that extends Stone's result for Boolean algebras (every Boolean algebra can be represented as a set algebra). This work of Jónsson and Tarski evolved from Tarski's purely mathematical study of the algebra of relations and includes no reference to modal logic or even logic in general. Jónsson and Tarski's theorem is a (more general) algebraic analog of Kripke's later semantic completeness results, yet this was not realized for some time. Not only was Tarski unaware of the connection, but it appears that both Kripke and Lemmon had not read the Jónsson and Tarski papers at the time in which they did their modal work in the late fifties and sixties, and Kripke claims to have reached the same result independently.

Lemmon (1966a and 1966b) adapts the algebraic methods of McKinsey to prove decidability results and representation theorems for various modal systems including T (though apparently in ignorance of Jónsson and Tarski's work). In particular, he extends McKinsey's method by introducing a new technique for constructing finite algebras of subsets of a Kripke model structure (discussed in the next section of this entry). Lemmon (1966b: 191) attributes to Dana Scott the main result of his second 1966 paper. This is a general representation theorem proving that algebras for modal systems can be represented as algebras based on the power set of the set K in the corresponding Kripke's structures. As a consequence, algebraic completeness translates into Kripke's model theoretic completeness. So, Lemmon elucidates very clearly the connection between Kripke's models whose elements are worlds and the corresponding algebras whose elements are sets of worlds that can be thought of as propositions, thereby showing that the algebraic and model theoretic results are deeply connected. Kripke (1963a) is already explicit on this connection. In The Lemmon Notes (1977), written in collaboration with Dana Scott and edited by Segerberg, the 1966 technique is transformed into a purely model theoretic method which yields completeness and decidability results for many systems of modal logic in as general a form as possible (1977: 29).

See also the SEP entry on the algebra of logic tradition. For a basic introduction to the algebra of modal logic, consult Hughes and Cresswell 1968, Chapter 17 on "Boolean Algebra and Modal Logic". For a more comprehensive treatment, see chapter 5 of Blackburn, de Rijke, and Venema 2001. See also Goldblatt 2003.

13.4 THE MODEL THEORETIC TRADITION

13.4.1 Carnap

In the early 1940s the recognition of the semantical nature of the notion of logical truth led Rudolf Carnap to an informal explication of this notion in terms of Leibnizian possible worlds. At the same time, he recognized that the many syntactical advances in modal logic from 1918 on were still not accompanied by adequate semantic considerations. One notable exception was Gödel's interpretation of necessity as provability and the resulting preference for S4. Carnap instead thought of necessity as logical truth or analyticity. Considerations on the properties of logically true sentences led him to think of S5 as the right system to formalize this 'informal' notion. Carnap's work in the early forties would then be focused on (1) defining a formal semantic notion of L-truth apt to represent the informal semantic notions of logical truth, necessity, and analyticity, that is, truth in virtue of meaning alone (initially, he drew no distinction between these notions, but clearly thought of analyticity as the leading idea); and (2) providing a formal semantics for quantified S5 in terms of the formal notion of L-truth with the aim of obtaining soundness and completeness results, that is, prove that all the theorems of quantified S5 are L-true, and that all the L-truths (expressible in the language of the system) are theorems of the system.

The idea of quantified modal systems occurred to Ruth Barcan too. In "A Functional Calculus of First Order Based on Strict Implication" (1946a) she added quantification to Lewis's propositional system S2; Carnap (1946) added it to S5. Though some specific semantic points about quantified modal logic will be considered, this entry is not focused on the development of quantified modal logic, but rather on the emergence of the model theoretic formal semantics for modal logic, propositional or quantified. For a more extensive treatment of quantified modal logic, consult the SEP entry on modal logic.

In "Modalities and Quantification" (1946) and in Meaning and Necessity (1947), Carnap interprets the object language operator of necessity as expressing at the object level the semantic notion of logical truth:

[T]he guiding idea in our constructions of systems of modal logic is this: a proposition p is logically necessary if and only if a sentence expressing p is logically true. That is to say, the modal concept of the logical necessity of a proposition and the semantical concept of the logical truth or analyticity of a sentence correspond to each other. (1946: 34)

Carnap introduces the apparatus of state-descriptions to define the formal semantic notion of L-truth. This formal notion is then to be used to provide a formal semantics for S5.

A state-description for a language L is a class of sentences of L such that, for every atomic sentence p of L, either p or ¬p, but not both, is contained in the class. An atomic sentence holds in a state-description R if and only if it belongs to R. A sentence ¬A (where A need not be atomic) holds in R if and only if A does not hold in R; (A∧B) holds in R if and only if both A and B hold in R, and so on for the other connectives in the usual inductive way; (∀x)Fx holds in R if and only if all the substitution instances of Fx hold in R. The range of a sentence is the class of state-descriptions in which it holds. Carnap's notion of validity or L-truth is a maximal notion, i.e., Carnap defines a sentence to be valid or L-true if and only if it holds in all state-descriptions. In later work Carnap adopts models in place of state-descriptions. Models are assignments of values to the primitive non-logical constants of the language. In Carnap's case predicate constants are the only primitive constants to which the models assign values, since individual constants are given a fixed pre-model interpretation and value assignments to variables are done independently of the models (1963a).

It is important to notice that the definition of L-truth does not employ the notion of truth, but rather only that of holding-in-a-state-description. Truth is introduced later as what holds in the real state description. To be an adequate formal representation of analyticity, L-truth must respect the basic idea behind analyticity: truth in virtue of meaning alone. In fact, the L-truths of a system S are such that the semantic rules of S suffice to establish their truth. Informally, state-descriptions represent something like Leibnizian possible worlds or Wittgensteinian possible states of affairs and the range of state-descriptions for a certain language is supposed to exhaust the range of alternative possibilities describable in that language.

Concerning modal sentences, Carnap adopts the following conventions (we use ϒ in place of Carnap's operator N for logical necessity). Let S be a system:

A sentence ϒA is true in S if and only if A is L-true in S (so a sentence ϒA is true in S if and only if A holds in all the state descriptions of S);

A sentence ϒA is L-true in S if and only if □A is true in S (so all state-descriptions agree in their evaluation of modal sentences).

From which it follows that:

ϒA is L-true in S if and only if A is L-true in S.

Carnap's conventions hold also if we substitute "truth in a state description of S" for "truth in S".

Carnap assumes a fixed domain of quantification for his quantified system, the functional calculus with identity FC, and consequently for the modal functional calculus with identity MFC, a quantified form of S5. The language of FC contains denumerably many individual constants, the universe of discourse contains denumerably many individuals, each constant is assigned an individual of the domain, and no two constants are assigned the same individual. This makes sentences like a=a L-true, and sentences like a=b L-false (1946: 49). Concerning MFC, the Barcan formula and its converse are both L-true, that is,

⊨(∀x)ϒFx↔ϒ(∀x)Fx.

This result is guaranteed by the assumption of a fixed domain of quantification. Carnap proves that MFC is sound, that is, all its theorems are L-true, and raises the question of completeness for both FC and MFC. Gödel proved completeness for the first order predicate calculus with identity, but the notion of validity employed was truth in every non-empty domain of quantification, including finite domains. Carnap instead adopts one unique denumerable domain of quantification. The adoption of a fixed denumerable domain of individuals generates some additional validities already at the pre-modal level which jeopardize completeness, for example "There are at least two individuals", (∃x)(∃y)(x≠y), turns out to be valid (1946: 53).

A consequence of the definitions of state-descriptions for a language and L-truth is that each atomic sentence and its negation turn out to be true at some, but not all, state-descriptions. Hence, if p is atomic both &p and &¬p are L-true. Hence, Lewis's rule of Uniform Substitution fails (if p∧¬p is substituted for p in &p we derive &(p∧¬p), which is L-false, not L-true). This is noticed by Makinson (1966a) who argues that what must be done is reinstate substitutivity and revise Carnap's naïve notion of validity (as logical necessity) in favor of a schematic Quinean notion ("A logical truth … is definable as a sentence from which we get only truths when we substitute sentences for its simple sentences" Quine 1970: 50) that will not make sentences like &p valid. Nonetheless, Carnap proves the soundness and completeness of propositional S5, which he calls "MPC" for modal propositional calculus, following Wajsberg. The proof however effectively employs a schematic notion of validity.

It has been proved that Carnap's notion of maximal validity makes completeness impossible for quantified S5, i.e., that there are L-truths that are not theorems of Carnap's MFC. Let A be a non-modal sentence of MFC. By convention (1), ϒA is true in MFC if and only if A is L-true in MFC. But A is also a sentence of FC, thus if L-true in MFC it is also L-true in FC, since the state descriptions (models) of modal functional logic are the same as those of functional logic (1946: 54). This means that the state descriptions hold the triple role of (i) first-order models of FC thereby defining first-order validity, (ii) worlds for MFC thereby defining truth for □A sentences of MFC, and (iii) models of MFC thereby defining validity for MFC. The core of the incompleteness argument consists in the fact that the non-validity of a first-order sentence A can be represented in the modal language, as ¬ϒA, but all models agree on the valuation of modal sentences, making ¬ϒA valid. Roughly, and setting aside complications created by the fact that Carnap's semantics has only denumerable domains, if A is a first-order non-valid sentence of FC, A is true in some but not all the models or state-descriptions. Given Carnap's conventions, it follows that ¬ϒA is true in MFC. But then ←ϒA is L-true in MFC, i.e., in MFC ⊨←ϒA. Given that the non-valid first-order sentences are not recursively enumerable, neither are the validities for the modal system MFC. But the class of theorems of MFC is recursively enumerable. Hence, MFC is incomplete vis-à-vis Carnap's maximal validity. Cocchiarella (1975b) attributes the result to Richard Montague and Donald Kalish. See also Lindström 2001: 209 and Kaplan 1986: 275–276.

13.4.2 Kripke's Possible Worlds Semantics

Carnap's semantics is indeed a precursor of Possible Worlds Semantics (PWS). Yet some crucial ingredients are still missing. First, the maximal notion of validity must be replaced by a new universal notion. Second, state-descriptions must make space for possible worlds understood as indices or points of evaluation. Last, a relation of accessibility between worlds needs to be introduced. Though Kripke is by no means the only logician in the fifties and early sixties to work on these ideas, it is in Kripke's version of PWS that all these innovations are present. Kanger (1957), Montague (1960, but originally presented in 1955), Hintikka (1961), and Prior (1957) were all thinking of a relation between worlds, and Hintikka (1961) like Kripke (1959a) adopted a new notion of validity that required truth in all arbitrary sets of worlds. But Kripke was the only one to characterize the worlds as simple points of evaluation (in 1963a). Other logicians were still thinking of the worlds fundamentally as models of first-order logic, though perhaps Prior in his development of temporal logic was also moving towards a more abstract characterization of instants of time. Kripke's more abstract characterization of the worlds is crucial in the provision of a link between the model theoretic semantics and the algebra of modal logic. Kripke saw very clearly this connection between the algebra and the semantics, and this made it possible for him to obtain model theoretic completeness and decidability results for various modal systems in a systematic way. Goldblatt (2003: section 4.8) argues convincingly that Kripke's adoption of points of evaluation in the model structures is a particularly crucial innovation. Such a generalization opens the door to different future developments of the model theory and makes it possible to provide model theories for intensional logics in general. For these reasons, in this entry we devote more attention to Kripke's version of PWS. For a more comprehensive treatment of the initial development of PWS, including the late fifties work on S5 of the French logician Bayart, the reader is referred to Goldblatt 2003. On the differences between Kanger's semantics and standard PWS semantics, see Lindström 1996 and 1998.

Kripke's 1959a "A Completeness Theorem in Modal Logic" contains a model theoretic completeness result for a quantified version of S5 with identity. In Kripke's semantic treatment of quantified S5, which he calls S5*=, an assignment of values to a formula A in a domain of individuals D assigns a member of D to each free individual variable of A, a truth value T or F to each propositional variable of A, and a set of ordered n-tuples of members of D to each n-place predicate variable of A (the language for the system contains no non-logical constants). Kripke defines a model over a non-empty domain D of individuals as an ordered pair (G,K), such that G∈K,K is an arbitrary subset of assignments of values to the formulas of S5*=, and all H∈K agree on the assignments to individual variables. For each H∈K, the value that H assigns to a formula B is defined inductively. Propositional variables are assigned T or F by hypothesis. If B is P(x1,…,xn), B is assigned T if and only if the n-tuple of elements assigned to x1, …, xn belongs to the set of n-tuples of individuals that H assigns to P.H assigns T to ¬B if and only if it assigns F to B.H assigns T to B∧C if and only if it assigns T to B and to C. If B is x=y it is assigned T if and only if x and y are assigned the same value in D. If B is (∀x)Fx it is assigned T if and only if Fx is assigned T for every assignment to x. □B is assigned T if and only if B is assigned T by every H∈K.

The most important thing to be noticed in the 1959 model theory is the definition of validity. A formula A is said to be valid in a model (G,K) in D if and only if it is assigned T in G, to be valid in a domain D if and only if it is valid in every model in D, and to be universally valid if and only if it is valid in every non-empty domain. Kripke says:

In trying to construct a definition of universal logical validity, it seems plausible to assume not only that the universe of discourse may contain an arbitrary number of elements and that predicates may be assigned any given interpretations in the actual world, but also that any combination of possible worlds may be associated with the real world with respect to some group of predicates. In other words, it is plausible to assume that no further restrictions need be placed on D,G, and K, except the standard one that D be non-empty. This assumption leads directly to our definition of universal validity. (1959a: 3)

This new universal notion of validity is much more general than Carnap's maximal validity. The elements H of K still correspond to first-order models, like Carnap's state-descriptions, and in each Kripke model the elements H of K are assigned the same domain D of individuals and the individual variables have a fixed cross-model assignment. So far the only significant divergence from Carnap is that different Kripke models can have domains of different cardinality. This by itself is sufficient to reintroduce completeness for the non-modal part of the system. But the most significant development, and the one that makes it possible to prove completeness for the modal system, is the definition of validity not as truth in all worlds of a maximal structure of worlds, but as truth across all the subsets of the maximal structure. The consideration of arbitrary subsets of possible worlds, makes it possible for Kripke's model theory to disconnect validity from necessity. While necessities are relative to a model, hence to a set of worlds, validities must hold across all such sets. This permits the reintroduction of the rule of Uniform Substitution. To see this intuitively in a simple case, consider an atomic sentence p. The classical truth-table for p contains two rows, one where p is true and one where p is false. Each row is like a possible world, or an element H of K. If we only consider this complete truth table, we are only considering maximal models that contain two worlds (it makes no difference which world is actual). By the definition of truth for a formula □B,□p is false in all the worlds of the maximal model, and □p is true in all of them. If validity is truth in all worlds of this maximal model, like for Carnap, it follows that ⊨□p, but in S5 ⊬□p. If instead we define validity as Kripke does, we have to consider also the non-maximal models that contain only one world, that is incomplete truth-tables that cancel some rows. Hence, there are two more models to be taken into consideration: one which contains only one world H=G where p is true, hence so is □p, and one which contains only one world H=G where p is false and so is □p as well as &p. Thanks to this last model ⊭□p. Notice that the crucial innovation is the definition of validity as truth across all subsets of worlds, not just the maximal subset. The additional fact that validity in a model is defined as truth at the actual world of the model—as opposed to truth at all worlds of the model—though revealing of the fact that Kripke did not link the notion of necessity to the notion of validity, is irrelevant to this technical result.

Kripke's completeness proof makes use of Beth's method of semantic tableaux. A semantic tableau is used to test whether a formula B is a semantic consequence of some formulas A1,…,An. The tableau assumes that the formulas A1,…,An are true and B is false and is built according to rules that follow the definitions of the logical connectives. For example, if a formula ¬A is on the left column of the tableau (where true formulas are listed), A will be put on the right column (where false formulas are listed). To deal with modal formulas, sets of tableaux must be considered, since if ϒA is on the right column of a tableau, a new auxiliary tableau must be introduced with A on its right column. A main tableau and its auxiliary tableaux form a set of tableaux. If a formula A∧B is on the right column of the main tableau, the set of tableaux splits into two new sets of tableaux: one whose main tableau lists A on its right column and one whose main tableau lists B on the right column. So we have to consider alternative sets of tableaux. A semantic tableau is closed if and only if all its alternative sets are closed. A set of tableaux is closed if it contains a tableau (main or auxiliary) that reaches a contradiction in the form of (i) one and the same formula A appearing in both its columns or (ii) an identity formula of the form a=a in its right side (this is an oversimplification of the definition of a closed tableau, but not harmful for our purposes). Oversimplifying once more, the structure of Kripke's completeness proof consists of proving that a semantic tableau used to test whether a formula B is a semantic consequence of formulas A1,…,An is closed if and only (i) in S5*= A1,…,An⊢B and (ii) A1,…,An⊨B. This last result is achieved by showing how to build models from semantic tableaux. As a consequence of (i) and (ii) we have soundness and completeness for S5*=, that is: A1,…,An⊢B if and only if A1,…,An⊨B.

The 1959 paper contains also a proof of the modal counterpart of the Löwenhein-Skolem theorem for first-order logic, according to which if a formula is satisfiable in a non-empty domain then it is satisfiable, and hence valid (true in G), in a model (G,K) in a domain D, where both K and D are either finite or denumerable; and if a formula is valid in every finite or denumerable domain it is valid in every domain.

Kripke's 1962 "The Undecidability of Monadic Modal Quantification Theory" develops a parallel between first-order logic with one dyadic predicate and first-order monadic modal logic with just two predicate letters, to prove that this fragment of first-order modal logic is already undecidable.

Of great importance is the paper "Semantical Analysis of Modal Logic I" (Kripke 1963a) where normal systems are treated. It is here that Kripke fully develops the analogy with the algebraic results of Jónsson and Tarski and proves completeness and decidability for propositional systems T, S4, S5, and B (the Brouwersche system), which is here introduced. Kripke claims to have derived on his own the main theorem of "Boolean Algebras with Operators" by an algebraic analog of his own semantical methods (69, fn. 2). It is in this paper that two crucial generalizations of the model theory are introduced. The first is the new understanding of the elements H of K as simple indices, not assignments of values. Once this change is introduced, the models have to be supplemented by an auxiliary function Φ needed to assign values to the propositional variables relative to worlds. Hence, while in the 1959 model theory there can be no two worlds in which the same truth-value is assigned to each atomic formula [which] turns out to be convenient perhaps for S5, but it is rather inconvenient when we treat normal MPC's in general (1963a: 69)

we can now have world duplicates. What is most important about the detachment of the elements of K from the evaluation function is that it opens the door to the general consideration of modal frames, sets of worlds plus a binary relation between them, and the correspondence of such frames to modal systems. So, the second new element of the paper, the introduction of a relation R between the elements of K, naturally accompanies the first. Let it be emphasized once again that the idea of a relation between worlds is not new to Kripke. For example, it is already present as an alternativeness relation in Montague 1960, Hintikka 1961, and Prior 1962, where the idea is attributed to Peter Geach.

In 1963a Kripke "asks various questions concerning the relation R" (1963a: 70). First, he shows that every satisfiable formula has a connected model, i.e., a model based on a model structure (G,K,R) where for all $H \in K$, $GR*H$, where $R*$ is the ancestral relation corresponding to R. Hence, only connected models need to be considered. Then, Kripke shows the nowadays well-known results that axiom 4 corresponds to the transitivity of the relation R, that axiom B correspond to symmetry, and that the characteristic axiom of S5 added to system T corresponds to R being an equivalence relation. Using the method of tableaux, completeness for the modal propositional systems T, S4, S5, and B vis-à-vis the appropriate class of models (reflexive structures for T) is proved. The decidability of these systems, including the more complex case of S4, is also proved. (For a more detailed treatment of frames, consult the SEP entry on modal logic.)

In the 1965 paper "Semantical Analysis of Modal Logic II", Kripke extends the model theory to treat non-normal modal systems, including

Lewis's S2 and S3. Though these systems are considered somewhat unnatural, their model theory is deemed elegant. Completeness and decidability results are proved vis-à-vis the proper class of structures, including the completeness of S2 and S3, and the decidability of S3. To achieve these results, the model theory is extended by the introduction of a new element N⊆K in the model structures (G,K,R,N).N is the subset of normal worlds, i.e., worlds H such that HRH. Another interesting aspect of the non-normal systems is that in the model theoretic results that pertain to them, G (the actual world) plays an essential role, in particular in the S2 and S3 model structures the actual world has to be normal. Instead, the rule of necessitation that applies to normal systems makes the choice of G model theoretically irrelevant.

The great success of the Kripkean model theory notwithstanding, it is worth emphasizing that not all modal logics are complete. For incompleteness results see Makinson 1969, for a system weaker than S4; and Fine 1974, S. Thomason 1974, Goldblatt 1975, and van Benthem 1978, for systems between S4 and S5. Some modal formulas impose conditions on frames that cannot be expressed in a first-order language, thus even propositional modal logic is fundamentally second-order in nature. Insofar as the notion of validity on a frame abstracts from the interpretation function, it implicitly involves higher-order quantification over propositions. On the correspondence between frame validity and second-order logic and on the model-theoretic criteria that distinguish the modal sentences that are first-order expressible from those that are essentially second-order see Blackburn and van Benthem's "Modal Logic: A Semantic Perspective" (2007a).

In 1963b, "Semantical Considerations on Modal Logic", Kripke introduces a new generalization to the models of quantified modal systems. In 1959 a model was defined in a domain D. As a result all worlds in one model had the same cardinality. In 1963b models are not given in a domain, hence worlds in the same model can be assigned different domains by a function Ψ that assigns domains to the elements H of K. Given the variability of domains across worlds, Kripke can now construct counter-examples both to the Barcan Formula

(∀x)ϒFx→ϒ(∀x)Fx

and its converse

ϒ(∀x)Fx→(∀x)ϒFx.

The Barcan formula can be falsified in structures with growing domains. For example, a model with two worlds, G and one other possible world H extending it. The domain of G is {a} and Fa is true in G. The domain of H is the set {a,b} and Fa, but not Fb, is true in H. In this model (∀x)ϒFx but not ϒ(∀x)Fx is true in G. To disprove the converse of the Barcan formula we need models with decreasing domains. For example, a model with two worlds G and H, where the domain of G is {a,b} and the domain of H is {a}, with Fa and Fb true in G,Fa true in H, but Fb false in H. This model requires that we assign a truth-value to the formula Fb in the world H where the individual b does not exist (is not in the domain of H). Kripke points out that from a model theoretical point of view this is just a technical choice.

Kripke reconstructs a proof of the converse Barcan formula in quantified T and shows that the proof goes through only by allowing the necessitation of a sentence containing a free variable. But if free variables are instead to be considered as universally bound, this step is illicit. Necessitating directly an open formula, without first closing it, amounts to assuming what is to be proved. Prior 1956 contain a proof of the Barcan formula

&(∃x)Fx→(∃x)&Fx.

Kripke does not discuss the details of Prior’s proof. Prior’s proof for the Barcan formula adopts Łukasiewicz’s rules for the introduction of the existential quantifier. The second of these rules states that if ⊢A→B then ⊢A→(∃x)B. Prior uses the rule to derive

⊢&Fx→(∃x)&Fx

from

⊢&Fx→&Fx.

This seems to us to be the ‘illegitimate’ step in the proof, since

&Fx→(∃x)&Fx

does not hold in a model with two worlds G and H, where the domain of G is {a} and the domain of H is {a,b}, and where Fa is false in both G and H, but Fb is true in H. In this model &Fx is true but (∃x)&Fx is false in G. In this counter-model &Fx is made true in G by the individual b that is not in the domain of G. In general, the rule that if ⊢A→B then ⊢A→(∃x)B does not preserve validity if we allow that Fx may be made true at a world by an individual that does not exist there. We conclude that the rule is to be rejected to preserve the soundness of S5 relatively to this model theoretic assumption.

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. Discuss the Syntactic Tradition.

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2. What are Matrix Method and Some Algebraic Results?

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3. Discuss the Model Theoretic Tradition.

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13.5 LET US SUM UP

Brief history and philosophical origins of modal logic.

Modes of truth, modalities and a spectrum of modal logics. Necessary and possible truths. Alethic modal logics.

Some important modal principles and systems of modal logic.

13.6 KEY WORDS

Syntactic Tradition: Syntactic Structures is a major work in linguistics by American linguist Noam Chomsky. It was a clear break with the existing tradition of language study.

Theoretic Tradition: It argues that that such as theory should not be thought of as a present, or perhaps even future, construction, but rather as a present device, or method, for thinking multiple traditions.

13.7 QUESTIONS FOR REVIEW

1. Discuss the Lewis Systems.
2. Discuss the Other Systems and Alternative Axiomatizations of the Lewis Systems.

13.8 SUGGESTED READINGS AND REFERENCES

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13.9 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. See Section 13.2
2. See Section 13.3
3. See Section 13.4

UNIT 14: THE LEWIS SYSTEM OF STRICT IMPLICATION

STRUCTURE

14.0 OBJECTIVES

After this unit 14, we can able to know:

- To know the Brief Biography Lewis
- To overview of Conceptual Pragmatism
- To highlight Logic and Language
- To discuss the A Priori and the Analytic
- To know about the Empirical Knowledge
- To discuss the concept of The Given
- To understand the Action, the Good, and the Right

14.1 INTRODUCTION

Clarence Irving (C.I.) Lewis was perhaps the most important American academic philosopher active in the 1930s and 1940s. He made major contributions in epistemology and logic, and, to a lesser degree, ethics.

Lewis was also a key figure in the rise of analytic philosophy in the United States, both through the development and influence of his own writings and through his influence, direct and indirect, on graduate students at Harvard, including some of the leading analytic philosophers of the last half of the 20th century.

14.2 BRIEF BIOGRAPHY

C.I. Lewis was born on April 12, 1883 in Stoneham, Massachusetts and died on February 2, 1964 in Menlo Park, California. He was an undergraduate at Harvard from 1902–1906, where he was influenced principally by the pragmatist, William James, and the idealist, Josiah Royce. Royce also supervised Lewis's 1910 Harvard Ph.D. dissertation " The Place of Intuition in Knowledge". While serving as Royce's teaching assistant in logic, Lewis read Whitehead's and Russell's Principia Mathematica, a book he both admired and criticized. Later, while teaching at the University of California at Berkeley from 1911–1920, his principal research interests switched to logic. Lewis wrote a series of articles on symbolic logic culminating in his 1918 monograph A Survey of Symbolic Logic (SSL) (Lewis 1918) in which he both surveyed developments in logic up to his day and concluded with his own modal system of strict implication. However, in response to criticism of his account of strict implication, Lewis deleted these sections from reprints of SSL and revised his treatment of their topics for his co-authored 1932 book Symbolic Logic (SL) (Lewis and Langford 1932) — "the first comprehensive treatment of systems of strict implication (or indeed of systems of modal logic at all)", according to Hughes and Cresswell (1968, 216).

Lewis returned to Harvard in 1920, where he taught until his retirement in 1953, becoming Edgar Peirce Professor of Philosophy in 1948. At Harvard, Lewis' major research interest switched back to epistemology. Starting with his much reprinted 1923 article, "A Pragmatic Conception of the A Priori" (Lewis 1923), he developed a distinctive position of his own which he labeled "conceptual pragmatism" and which he presented in a systematic way in his 1929 book Mind and the World Order (MWO) (Lewis 1929). MWO established Lewis as a major figure on the

American philosophical scene. In the 1930s and 1940s, partly in response to the challenge of positivism, the form and focus of Lewis' views changed, and, arguably, in subtle ways, some of the substance. In his 1946 book Analysis of Knowledge and Valuation (AKV), based on his 1944 Carus lectures, Lewis (1946) provided a systematic and carefully analytic presentation of his mature philosophical views. The first two thirds of the book consist of a thorough refinement and more precise presentation of his theory of meaning and of his epistemological views, and the last third consists of a presentation of his theory of value.

After retirement from Harvard, Lewis taught and lectured at a number of universities, including Princeton, Columbia, Indiana, Michigan State, and Southern California, but principally at Stanford. His 1954 Woodbridge Lectures at Columbia and 1956 Mahlon Powell Lectures at Indiana resulted in two last short books in ethics, The Ground and the Nature of the Right (Lewis 1955) and Our Social Inheritance (Lewis 1957). Lewis was the subject of a posthumously published “Library of Living Philosophers” volume (Schilpp 1968), an honour that indicates his standing in and perceived significance for American philosophy in the 1950s.

In his over thirty years at Harvard, Lewis taught some of the most eminent American philosophers of the last half of the twentieth century as graduate students, including W.V. Quine, Nelson Goodman, Roderick Chisholm, Roderick Firth, and Wilfrid Sellars. Although only Chisholm and Firth of these five were supervised by Lewis, and Sellars left Harvard without graduating, all five refer occasionally to Lewis in their writings, usually critically, and their own views sometimes developed in reaction to his. (Baldwin 2007 has an excellent discussion of the influence of Lewis on Quine, and of Lewis's philosophy generally.)

14.3 OVERVIEW OF CONCEPTUAL PRAGMATISM

In MWO, Lewis (1929, Chp. 1) argued that the proper method of philosophy isn't transcendental but rather reflective. Philosophy seeks the criteria or principles of the real, the right, the beautiful, and the logically valid that are implicit in human experience and activity.

Lewis (1929, 37–8) thought that, on reflection and analysis, we can distinguish three elements in perceptual knowledge: (1) the given or immediate data of sense, (2) the act of interpreting the given as an experience of one sort of thing as opposed to another, and (3) the concept by which we so interpret the given by relating it to other possibilities of experience. Our experience of the real is not given to us in experience but is constructed by us from the data of sense through acts of interpretation. So when I know that I am looking at a table and reflect on my experience, I realize, on analysis, that there are certain highly specific sensuous qualities presented to me that I am immediately aware of, and that, in the light of this and other experiences I recall, I expect that I would likely have certain other experiences, e.g., those of feeling something apparently hard, were I to have the experience of performing certain acts, e.g., reaching out with my hand. In doing so, it is the concept of seeing a table that I am applying to my experience rather than that of seeing a horse or that of hallucinating a table, either of which would have involved different expectations of experience consequent upon action. Only an active being can therefore have knowledge, and the principal function of empirical knowledge "is that of an instrument enabling transition from the actual present to a future which is desired and which the present is believed to signalize" (Lewis (1946), 4).

Statements expressing our beliefs about reality are translatable into, and thus entail and are entailed by, an indefinitely large set of counterfactual statements about what experiences we would have or would be likely to have, were we to be presented with certain sensory cues and were we to carry out further tests (Lewis 1929, 142; 1946, 180, 208). Objectively, what we actually experience may depend on the physical circumstances of perception, e.g. lighting conditions, and the bodily actions we perform, e.g. moving our eyes, as well, of course, as the character of objects in our environment. However, what matters ultimately for the meaning and confirmation of statements about objective reality, as Lewis makes clear in AKV, are only the "felt experience" of action and the directly presentable sense experiences contained in sensory cues and forming the experiential circumstances of action (Lewis 1946, 178-9, 245-6). The idea of a reality neither confirmable nor disconfirmable in principle by experience was thus for Lewis without meaning. What distinguished his view, he thought, from the superficially similar verification principle of meaning of the logical positivists was his emphasis on the mediating role of agency (Lewis 1941a, in Lewis 1970, 94).

Whether our objective beliefs are true depends on their verifiable or confirmable implications for future possible experience. However, in order to guide action effectively now while saving us from the hazards of action without foresight, empirical belief and its expectations for experience must be rationally credible (justified, warranted) now, antecedent to future verification (Lewis 1946, 254-7). Justification, as opposed to verification, is the focus of AKV much more so than MWO. Nonetheless, throughout Lewis' career from MWO to the end, there are common claims. First, empirical knowledge (rationally credible, justified, or warranted belief) is probable knowledge or belief. Second, probability is a logical or inferential relation between a conclusion or a belief and its premises or grounds. Third, the ultimate premises or grounds, as opposed to more immediate or proximate ones, relative to which a conclusion or belief is probable cannot themselves be probable but must be certain (Lewis 1929, 328-9, 340-1; Lewis 1946, 186-7; Lewis 1952a).

The direct apprehension of immediately given sense presentations, and statements expressing them, are incorrigible, indubitable, not in need of verification, and not subject to error, and so, in these various (and distinct) senses "certain". (For a useful discussion of senses of "certainty" in Lewis, see Firth 1964 and Firth 1968 in Schilpp 1968.) However, with no possibility of error or incorrectness to contrast with the immediate awareness of the given, Lewis decides the normative label "knowledge" shouldn't really be applied to it. Our objective interpretations of experience, on the other hand, are not only fallible—given and recalled experience doesn't guarantee the satisfaction of our expectations about future experience--but are always subject to revision in the light of action and further experience. Past experience and our recollection of it play a key role in the credibility of these interpretations. In MWO, Lewis (1929, 337) says that memory itself is an interpretation of given present recollection, and, as such, probable knowledge with testable or verifiable implications for future experience. However, this doesn't explain what warrants the interpretation and its expectations, a lacuna AKV corrects. Knowing for Lewis occurs in a non-instananeous "epistemological present" of sense presentation embedded in a mass of recollections or sense of past experience (Lewis 1946, 331). What is given and indubitably certain in this present are these sense presentations and these rememberings (Lewis 1946, 354, 362)), but what we recall of past experience is prima facie and non-inductively credible for us, just because so ostensibly remembered or recalled (Lewis 1946, 334), and thus can serve to make our expectations of future experience rationally credible as well.

Despite their lack of theoretical certainty, the beliefs we form by applying concepts to experience may count as knowledge so long as they are true and sufficiently warranted or justified. The members of a set of beliefs that already have some degree of confirmation or antecedent probability in relation to present and past experience may become even more credible if the antecedent probability of any one would be increased by assuming the others as given (Lewis 1946, 187, 338, 349, 351, 352). The congruence of a mature system of beliefs which exhibits this complex interlocking pattern of probability relations to each other and to experience helps to explain how many of these beliefs can rise to the standards for knowledge and be "practically certain" enough to be counted on in action.

Ordinary beliefs and interpretations, including perceptual beliefs, are, for Lewis, typically the product of habit or association in which we are guided by the elements of the given in the epistemological present but rarely if ever attend to them. Nonetheless, the justification of these beliefs as rationally credible requires that there be an inferential or logical relationship of a belief or a statement of it to grounds or reasons in experience that constitute evidence, largely inductive, for it (Lewis 1946, 315, Lewis 1952a, reprinted in Lewis 1970, 326). For Lewis, "the critical question for the validity of empirical knowledge is not whether grounds sufficient for the justification of the belief are actually contained in the explicit psychological state of the believer, but whether the knower's situation in empirical belief is such that sufficient grounds could be elicited upon inquiry, or whether it is such that this is even theoretically impossible" (Lewis 1946, 330). Restricting the term "knowledge" to cases where grounds are explicit would "be so rigoristic as to exclude most, if not all, of our attempted cognitions and would obscure the important distinction of practically valuable knowledge from 'ignorance' and from 'error'" (Lewis 1946,330), just as restricting the term to what we are certain of would. Nor should we think that the act of reflecting on and eliciting what in the epistemological present justifies our beliefs loses or takes us beyond the epistemological present (Lewis 1946, 330-2), and thus is an impossible ideal.

In MWO Lewis argues that probable empirical knowledge requires, on pain of infinite regress, some a priori knowledge of analytic principles explicating our concepts, the logical relationships among them, and the criteria for applying them to experience and determining what is real and what isn't real. Moreover, this knowledge must be "more than probable" and "certain", which suggests that they have a degree of warrant greater than the degree of probability empirical considerations, could yield (Lewis 1929, 311-12, 317,321). In AKV, he also says that mathematical and logical cognitions "may be certain" (Lewis, 1946, 29), at least in certain cases, but he worries how they can refer to anything beyond the cognition itself, and be classified as knowledge, if their truth is simply the unthinkability of their falsity, given our concepts or way of thinking. The answer is that we can be mistaken about what is implicit in our concepts. We can fail to observe what is implicit in our concepts and what their adoption consistently commits us to in our thinking, and that contrast with error allows him to classify a priori apprehension as knowledge. Nonetheless, any of these mistakes is "intrinsically possible of correction merely by taking thought of the matter" (Lewis 1946, 155) without empirical investigation. The correctness of the principles governing our concepts can be known a priori, independently of confirmation in experience, in so far as they can be certified or assured simply by analysis of meaning or reflection on the content of our concepts and our explicative principles (Lewis 1946, 151,165). So, quite apart from issues about certainty, the degree to which we are warranted in our a priori apprehensions needn't correspond to and isn't a function of their probability on the total set of empirical considerations (cf. Lewis 1926, reprinted in Lewis 1970, 243-4).

What then is tested and confirmed or disconfirmed by experience is the interpretation of experience in the light of our concepts, ordinary empirical concepts like dog as well as more abstract categories like causality or the concepts of logic. What isn't tested by experience is the validity of the concepts themselves, or the logical relationships amongst them, or the principles for applying them. Agents bring them to experience and the only criteria they answer to are pragmatic ones of utility or convenience (Lewis 1929, 271–2).That implies that they are also revisable on pragmatic grounds, as Lewis himself recognizes he is doing to some extent with the concept of knowledge itself (Lewis 1946,27-29, 183) .

Right at the heart of Lewis' philosophical system, then, are several theses that weren't original to Lewis, but the critical discussion (and sometimes rejection) of which, often in the form Lewis gave to them, was central to much analytic philosophy in the last half of the twentieth century. Among them are: (1) a sharp analytic/synthetic, a priori/ a posteriori distinction, (2) reductionism concerning the meaning of a physical object statement to the actual and possible sense experiences that would confirm the statement, (3) a foundation for all empirical knowledge in our direct apprehension or immediate awareness of the given character of experience and our recollections of it, and (4) the division of experience into its given content or character, on the one hand, and the form we impose on it, or the concepts in the light of which we interpret it, on the other. (Quine (1953) famously called (1) and (2) the “two dogmas of empiricism”; Sellars (1963) called (3) the “myth of the given”; and Davidson (1984) called (4) the “third dogma of empiricism”, although in Lewis' mind (4) may owe more to Kant—on whom Lewis taught a course regularly at Harvard—than to the empiricists.)

At the same time, Lewis (1946, 9–11, 254–9) also laid down a framework of assumptions, most explicitly in AKV, within which analytic epistemology flourished in the last half of the 20th century: (1) knowledge is sufficiently justified (warranted, rationally credible) true belief , (2) a belief may be justified without being true and true without being justified, and (3) epistemology seeks to elicit criteria or principles of justification or rationally credibility.

14.4 LOGIC AND LANGUAGE

Lewis was dissatisfied with the extensional truth functional logic of Principia Mathematica, and with its understanding of implication as material implication, according to which the truth of the if-then conditional p ⊃ q expressing the material implication of q by p is a function just of the truth or falsity of p and q. p ⊃ q is equivalent to ~(p & ~q) and is true just in case it isn't the case both that p is true and q is false. As a result, among the theses of Principia Mathematica are p ⊃ (q ⊃ p) and not-p ⊃ (p ⊃ q). In other words, a true proposition, whatever it happens to be, is implied by any proposition whatsoever, true or false, and a false proposition, whatever it happens to be, implies any proposition whatsoever, true or false. Lewis didn't deny these theses, properly understood relative to the definition of material implication. However, he did think that these so-called “paradoxes of material implication” meant that material implication doesn't provide a proper understanding of any ordinary notion of implication, according to which one proposition implies another just in case the latter logically follows from and is deducible from the former.

To explicate this notion he defined strict implication, according to which the if-then conditional p strictly implies q expressing the strict implication of q by p is equivalent to ~◊(p & ~q), and is true just in case it is not possible that p is true and q is false. Strict implication is an intensional notion, and the logic of strict implication is a form of modal logic. The system of strict implication developed in SSL (Lewis 1918) was distinguished from earlier modal logics by its axiomatic presentation in the light of the work of Whitehead and Russell. However, Lewis faced a number of criticisms, including one by Emil Post that one of Lewis' postulates led to the result that it was indeed impossible that p just in case it was false that p, so that Lewis' SSL system reduced to an extensional one. Lewis (Lewis and Langford 1932) eliminated these problems in SL and provided distinct systems of strict implication or modal logic, S1–S5, each stronger than its predecessor (with S3 the system of SSL). S1 contained the following axioms:

(p & q) strictly implies (q & p)
(p & q) strictly implies p
p strictly implies (p & p)
((p & q) & r) strictly implies (p & (q & r))
((p strictly implies q) & (q strictly implies r)) strictly implies (p strictly implies r)
(p & (p strictly implies q)) strictly implies q
S2 adds to S1 the consistency postulate

◊(p & q) strictly implies ◊p,
which allows one to show that if p strictly implies q is a theorem, then so is ~◊~p strictly implies ~◊~q, i.e., ϒp strictly implies □q, expressing the strict implication of the necessity of q by the necessity of p. S3 adds to S1 the postulate

(p strictly implies q) strictly implies (~◊q strictly implies ~◊p)
S4 adds to S1 the iterative axiom:

~◊~p strictly implies ~◊~~◊~p, i.e.,
ϒp strictly implies ϒϒp
S5 adds to S1 the iterative axiom:

◊p strictly implies ~◊~◊p, i.e.,
◊p strictly implies ϒ⊒p
Critics objected that strict implication posed its own alleged paradoxes. Within Lewis' systems S2–S5, a necessarily true proposition is strictly implied by any proposition whatsoever, and a necessarily false proposition strictly implies any proposition whatsoever. However, Lewis (Lewis and Langford 1932) replied in SL that these alleged paradoxes are simply the result of entirely natural assumptions about valid deductive inference and entailment quite apart from the systems of strict implication, and thus are not a problem for the claim that strict

implication provides an explication of deducibility and entailment. (The presentation in this and the previous two paragraphs owes much to the excellent account and discussion of Lewis' systems of strict implication in Hughes and Cresswell 1968, Chapters 12–13).)

Lewis thought that there are an unlimited number of possible systems of logic. One example is the extensional propositional calculus of Principia according to which there are two truth values, true and false; other examples are the various systems of many valued logic that Lewis surveyed in SL, and, of course, Lewis' own various modal systems S1–S5. Lewis thought that that each of these systems is valid so long as it is internally consistent. The principles of the various alternatives simply define the meaning of the logical concepts and operators such as negation, truth/falsity, disjunction, implication, and thus they are all true (Lewis 1932, in Lewis 1970, 401). Bivalent systems simply have a different notion of truth and falsity from non-bivalent ones. Nonetheless, some systems may accord better than others with notions of truth or implication or deduction that are implicit in our everyday reasoning. Logics can thus be assessed pragmatically by their sufficiency for the guidance and testing of our usual deductions, systematic simplicity and convenience, and accord with our psychological limitations and mental habits. However, Lewis denied that he was claiming that principles of logic could be true without being necessarily true, or necessarily true without being necessarily necessary. A logic in which $\Box p$ strictly implies $\Box\Box p$ holds simply operates with a different notion of necessity from one in which it doesn't.

Lewis (1946, Chps. 3, 6) distinguished several modes of meaning in AKV. The denotation of a term is the class of actual things to which the term applies and is distinct from the comprehension — the class of possible or consistently thinkable things to which it applies. The signification of a term is the property the presence of which in a thing makes the term applicable, and the intension or connotation of a term is what is applicable to any possible thing to which the term is applicable. Intension can be linguistic intension or meaning, in which case it is the conjunction of terms applicable to any possible thing to which the term is applicable and thus substitutable for the term salva veritate, but since definitions must have criteria of application and these must ultimately be non-circular, the more basic dimension of intension is sense meaning. Sense meaning is the criterion in mind in terms of sense experience for classifying objects and applying a term, a schema or rule that speakers have in mind whereby a term applies to an actual or thinkable thing or signifies some property, and which would exist even if there were no linguistic expression for it.

Since linguistic intension is implicitly holistic and verbal definition eventually circular, Lewis (1929, 107) said in MWO that logical analysis isn't reduction to primitive terms, but is a matter of relating terms to each other. Concepts consist in relational structures of meaning. They require criteria of application in experience and the total meaning of a term for an individual consists of the concept it expresses and the sensory criteria for its application. Yet, the latter needn't be identical across individuals for there to be common concepts, Lewis argued (1929, 115). Instead, common concepts just require shared structures of linguistic definition and common or congruent modes of behaviour, in particular co-operative behaviour that is guided by these concepts, a social achievement of a common world that Lewis thought our community of needs and interests produces. One problem with this suggestion was pointed out by Quine (1960) in Word and Object with his indeterminacy of translation argument. From the standpoint of an interpreter, there can be alternative translation manuals or schemes that are consistent with the total set of a speaker's verbal and other behavioural dispositions. This is a problem that Lewis (1946, 144–5, 164) may have been aware of in AKV. In any case, in AKV he seems to draw back from the discussion of common concepts in MWO and to rest content with pointing out that any attribution of linguistic meaning or sense meaning to another is inductive and thus only probable, and any attribution of linguistic or sense meaning similar to ours is likewise inductive, fallible, and problematic.

Lewis (1946, 84–5, 93–5) drew a sharp distinction between analytic and synthetic truth. Analytic (or analytically true) statements are true by virtue of the definition of the terms they contained, and have zero intension (and universal comprehension). They are necessarily true, true in all possible worlds, no matter what else might be true of a world or thing, and yet are not equivalent in meaning to each other only due to the distinct intensions of their constituents. In MWO Lewis occasionally claimed that we create necessary truth by adopting concepts and criterial principles; in AKV he was more circumspect. It is a matter of convention or legislation that a term has the meaning it does, including sense meaning, but Lewis (1946, 155–7) denied that analytic truth was truth by convention. “A dog is an animal” is analytically true by virtue of the sense meaning of “dog” and “animal”, in particular, the inclusion of the criterion for applying the latter in the criterion for applying the former, and that isn't a matter of convention. However, Lewis never tried to define such inclusion further. Quine (1953) explicitly criticized Lewis and the analytic/synthetic distinction in “Two Dogmas of Empiricism”, and would have objected to the idea of resting the analytic/synthetic distinction on an undefined notion of meaning inclusion. Lewis (1946, 154), on the other hand, thought that meaning inclusion is as unproblematic and recognizable a fact as the inclusion of one plan in another, e.g., a plan to visit France in a plan to visit Paris, and didn't need further explication. Nonetheless, taking meaning inclusion to be a primitive fact also makes it more difficult to distinguish Lewis' analytic necessity from the rationalists’ synthetic necessity, despite his (Lewis 1946, 157) vigorous rejection of the latter. This is especially so since Lewis (1946, 129) denied that analytic truth is usefully elucidated as one that is reducible to logical truth with the substitution of definitions. For Lewis, the adequacy of a definition itself is a matter of analytic truth and what makes a truth a logical truth is that it is an analytic truth concerning certain symbols.

14.5 THE A PRIORI AND THE ANALYTIC

Lewis (1946, 29–31) thought that necessary truths are knowable a priori, independently of experience. In applying concepts like those of red or apple to current experience, and so interpreting experience, we form expectations and make predictions about future experience, conditional on actions we might perform. Our beliefs constitute empirical knowledge in so far as past experience gives us good reason (largely inductive) for making these predictions. However, we aren't making predictions about future experience simply in stating what these concepts are, and what their definitions are, and what defining criteria they provide for applying them to experience. Such statements are explicative, not predictive, and so neither falsifiable by failed prediction nor verifiable by successful prediction nor justified by inductive evidence. The a priori is what we are not required to abandon, no matter what experience may bring, and in that sense, true no matter what, and in that sense necessary (Lewis 1929, 267.) However, a priori principles are neither principles that are universal nor ones that we have to accept. The acceptance of a set of concepts is a matter of decision or legislation or the adoption of an intention to employ certain criteria in the interpretation of experience, something for which there are alternatives, but for which the criteria are not empirical but pragmatic.

In MWO, Lewis (1929, 254) also thought that the a priori extended to fundamental laws of nature defining basic concepts like mass or energy or simultaneity, and thus included some of what are typically regarded as the basic principles of a scientific theory. Further, besides criteria like convenience and conformity to human bent, pragmatic considerations mentioned in MWO (Lewis 1929, 267) include factors like intellectual simplicity, economy, comprehensiveness, and thus the overall achievement of intellectual order. However, unlike Wilfrid Sellars and many others in the latter half of the 20th century, Lewis never recognized such factors as criteria of empirical justification. The reason seems to be that Lewis (1936b, reprinted in Lewis 1970, 286) didn't think that these factors make a hypothesis any more probable, in contrast, presumably, to conformity to standard criteria of induction: “What such simplicity and convenience determine is not truth or even probability but merely simplicity and convenience, which have their reasonable place in the choice of working hypotheses when no more decisive criterion is presently at hand”. At the same time, he thought that the acceptance and rejection of scientific theories wasn't entirely empirical. The choice of a system of concepts and a priori explicative principles to apply to experience and interpret it in the light of is determined by pragmatic considerations, not truth or probability. The simpler set of scientific concepts and explicative principles is no more true or likely to be true than the simpler logic is more true or valid. Moreover, pragmatic considerations might lead us in the face of experience to abandon our scientific concepts and a priori principles explicating them without making the latter empirically unwarranted or any less a priori warranted.

Empirical belief and a priori belief aren't logically separated but intertwined for Lewis. The empirical belief that there are no unicorns presupposes the concepts of negation and unicorn, and thus more general a priori principles governing negation and explicating the concept of unicorn, e.g. unicorns are horse like creatures with a horn in their nose. Repeatedly failing to apply the concept of unicorn successfully to experience may make it extremely likely that there are no unicorns and eventually lead us to drop the concept altogether from our conceptual repertoire as useless clutter, along with beliefs explicating the concept, but it doesn't do so by disconfirming or making any less likely the belief that if anything is a unicorn, it is a horned horse-like creature. More important cases Lewis discusses are ones where we discover that there are no Euclidean figures in our space and cease employing Euclidean geometry to interpret experience, or ones where cruder categories for interpreting experience are replaced by more fine grained ones that carve up experience in novel ways that are more valuable for our purposes, or ones where inventions open up the bounds of experience and lead us to abandon an old theory that can accommodate such experience but in a less simple way than a novel one. Categories for Lewis don't really change or alter but are given up and replaced, and old truths (as opposed to falsehoods) are replaced by new ones, not contradicted by them (Lewis 1929, 267-8).

The most radical challenge to Lewis came from Quine (1953) who argued that the distinction between so called a priori truths and a posteriori truths is just one of degree. The argument has two steps. First, empirical hypotheses have implications for experience only in conjunction with various empirical generalizatons and other background assumptions, e.g., about the circumstances of perception. Recalcitrant experience thus tells us only that some belief or assumption in the total set that implies a contrary experience is false, not which one, and thus any statement can be held true, no matter what experience brings, so long as we make enough adjustments to the rest of our beliefs and assumptions. Second, empirical hypotheses can't logically imply anything about experience except against a background of assumed laws of logic. Recalcitrant experience can, in principle, then, lead us to revise an assumed logical principle in our web of belief rather one of our other beliefs.

With respect to the second stage, some philosophers might object that logic is part of the framework within which beliefs have logical implications and can't be part of the same system of belief itself. However, Lewis himself might have trouble with this suggestion, since he recognized the possibility of alternative logics, and presumably, any decision, even pragmatic, about the adoption or rejection of a logic must operate on some logical assumptions. In any case, Lewis himself recognizes that in principle experience could lead us to abandon logical beliefs and replace them with others. What he will deny is that it does so by making these principles empirically improbable, and thus any less a priori warranted. Arguably, he may be assuming an excessively narrow view of what makes for probability. With respect to the first stage, Lewis (1946) in AKV will deny Quine's assumption that objective statements never entail conditionals about experience without supposing other objective statements true and his assumption that the antecedents of these conditionals are never themselves certain, as we shall see in the next two sections. Moreover, systems of objective hypotheses, despite their various interconnections, aren't tested as a “block”, but have separable and distinct probabilistic connections to others (but not thereby to all) and to experience establishing differing antecedent probabilities and degrees of confidence, in the light of which the relevance of tests for the various hypotheses needs to be assessed differentially. (Lewis, 1936b, reprinted in Lewis 1970, 285-6, Lewis 1946, 349-52). However, even so, there may be room for alternative ways of modifying systems of hypotheses and their degrees of credibility in the light of experience, and different responses to probabilities that are unsettled and may change with future testing. So, in an unpublished lecture Lewis (1936b, 282-7) says we are left with pragmatic factors like economy, convenience, simplicity, or making the least alteration to beliefs, at least for the time being, to choose "working hypotheses" even if they don't count as empirical knowledge. Lewis (1946) omits this discussion from AKV. Nonetheless it highlights the need for Lewis to provide a positive account of non-empirical, non-inductive a priori knowledge of principles explicating our concepts and how we have it, not just a negative account that simply contrasts it with empirical knowledge.

In MWO, he says the a priori is knowable by the reflective and critical formulation of our own principles of classification, a least with respect to meaning connections explicitly before the mind (Lewis 1929, 287-8) and in AKV that a priori truths are certifiable by reference to meanings alone (and their relations like inclusion), and tested simply by what we can think or imagine being so. (Lewis 1946, 35, 151-3) However, the only explanation he gives of why this should warrant us in thinking anything necessary or possible is that "as what we intend at the moment at least, a meaning seems to be as open to direct examination as anything we are likely to discover" (Lewis 1946, 145). It is this that warrants us as dismissing any apparent Euclidean triangle the sum of whose angles isn't 180 degrees as either a mismeasurement or not a Euclidean triangle rather than a counterexample to Euclidean geometry. Barring certainty, there seems to be a half-acknowledged, non-inductive, basic principle of a priori credibility assumed here, to the effect that if on reflection on our concepts and meanings and classificatory intentions, we think A includes B, then we are at least prima facie warranted in thinking so. In the next section, we shall see that Lewis isn't in principle averse to non-inductively supported principles of prima facie warrant.

14.6 EMPIRICAL KNOWLEDGE

In AKV, Lewis distinguished three classes of empirical statements. First, there are expressive statements formulating what is presently given in experience and about the truth of which we can be certain (Lewis 1946, 171-71, 183, 204, 327). Second, there are terminating judgements and statements formulating and predicting what we would experience were we to be presented with some sensory cue and perform some action. The form of terminating judgements is:

If (or given) S, then if A, it would be the case that E, i.e. ((S & A) → E), (Lewis 1946, 184, 205)

where S, A, and E all are formulated in expressive language and concern particular presentable experiences about which we can be certain, and "→" is neither logical entailment nor material implication but what Lewis called "real connection" that gives rise to subjunctive or counterfactual conditionals. Real connections (an example of which are causal connections) are inductively established correlations by virtue of which one observable item may indicate another. Terminating judgements, as expressing a general claim about repeatable mode of action and a sequent experience, Lewis (1946, 219) claimed, are not decisively verifiable but are decisively falsifiable. Third, there are non-terminating or objective judgments that are confirmable and disconfirmable by experience, thanks to their sense meaning, but are neither decisively verifiable nor decisively falsifiable.

Objective judgements include not only perceptual judgements like "There is a white piece of paper before me" in which we conceptualize and interpret a given experience by relating it to other possible experiences, but also a vast number of other beliefs about the material world supported by our perceptual beliefs, e.g., statements about the future outcome of space explorations, or generalizations like "All men have noses", or non-analytic statements about theoretical entities. Objective judgements don't strictly imply terminating judgements of the form (S & A) → E (Lewis 1946, 219). Instead, the sense meaning of objective judgments consists of an indefinitely large set of general conditional probability judgements of the form "If it were the case that S & A, then, in all likelihood, E" (Lewis, 1946, 237). Any objective perceptual judgement P thus analytically entails and is entailed by, an indefinitely large set of hypothetical or conditional judgements of the form,

(S & A) → (h)E,

where (h)E means that, in all probability E (Lewis 237-53),

None of these conditional judgements are decisively verifiable or falsifiable by experience. (Lewis (1946, 247) calls the statements that constitute the empirical content of objective statements terminating judgements nonetheless). However, as expressing real connections, they are nonetheless confirmable and disconfirmable by experience, as are the objective judgements whose sense meaning they constitute.

For example, to use Lewis's examples, suppose P is “a sheet of paper lies before me”. Then, its analytic entailments might include, “If S1 (I were to seem to see a sheet of paper before me) and A1 (I were to seem to move my eyes), then, probably, E1 (I would seem to see the sheet of paper displaced)”, as well as “If S2 (I were to seem to feel paper with my fingers), and A2 (I were to seem to pick it up and tear it), then, probably, E2 (I would seem to see or feel torn paper)”, and so on. On the other hand, suppose P is “There is a doorknob before me”. Its truth might then entail the truth of a complex set of conditionals like “If I were to seem to see a doorknob, and were to seem to reach out towards it and grasp it, then, probably, I would seem to feel something hard and round”, etc.

In MWO, Lewis says in one place not only that appearances are physically conditioned by objects and the physical circumstances of one's body and perception, which is certainly a reason for wanting to know such physical facts, but that these conditions enter into the basic understanding or meaning of material object statements: “It is such conditions which are expressed in the ‘if’ clause of those ‘If … then …’ propositions in which the predictions implicit in an interpretation may be made explicit” (Lewis 1929, 286). However, in AKV he explicitly rejects this position: “Thus those conditions which are directly pertinent to a confirmation and genuinely ascertainable are not objective facts but must be included amongst the given appearances. They must be items of direct presentation; and we might think of them as already covered by ‘S’ in our paradigm: S being given, if A then, with probability M, E” (Lewis 1946, 246). The result is that S in Lewis' paradigm strictly speaking won't just include the visual presentation of a doorknob, for example, but the appearance of daylight or the feeling of being clear headed as part of the whole presentation. Arguably, that, together with the probability qualifier, allows him to avoid Quine's worries about circumstances of perception and the testable implications of empirical beliefs (Lewis 1946, 242-6). However, it also means that S will enter into the evidence for a great many objective beliefs (Lewis 1946, 246) that won't, therefore, be entirely probabilistically independent.

No number of successful or failed tests will render the objective judgement true or false with theoretical certainty. However, Lewis thought the principle of inverse probabilities meant that the judgement can be highly probable with a few positive confirmations, even practically certain, in so far the probability of P when S and A and E obtain may approach certainty as the improbability of E approaches certainty when S and A and not P obtain. The principle also explains why further tests may increase our warranted assurance in the judgement even more, though not as dramatically as earlier tests increased our warrant (Lewis 1946, 190-92). Since confirming and gaining assurance that P gives us assurance in all the predictions that P entails about future experience (Lewis 1946, 239), the principle of inverse probabilities may explain how we can act on these predictions with increasing confidence. Even though experience of instances of S and A and E can confirm and increase the probability of one sensory conditional entailed by P independent of experiential confirmation of the other sensory conditionals entailed by P , experiential confirmation of one conditional increases the probability of the others, and vice versa (Lewis 1946, 348, footnote 6).

Our empirical knowledge of objects and objective events and properties, the generalizations they support concerning patterns of objective events and properties, and the use we make of all this for further inductions, has a complex “many-storied character”. Nonetheless, the “whole edifice still rests at bottom on these primitive generalizations which we make in terms of direct experience” (Lewis 1946, 261). (Lewis (1929, 332; 1946, 361) contrasted these primitive generalizations that underlie our objective beliefs with what he says we normally call empirical generalizations that concern patterns of objective events and may formulate natural laws supporting causal explanations.) However, the empirical justification for these primitive generalizations and ultimately for our objective beliefs can't rest on current sense experience alone and requires evidence concerning the past. At the same time, what is given to us isn't the past itself about which we can never be certain, but just current sense presentation and current recollection or sense of past experience.

Lewis appreciated the problem memory posed for his epistemology much more clearly in AKV than in MWO. In AKV, Lewis (1946, 334) argued that whatever we ostensibly remember, whether as explicit recollection or merely in our sense of the past, is prima facie credible just because so remembered. So there are data of sense presentation and also data of “seeming to remember” or “present memory” (Lewis 1946, 353,354) that constitute our ultimate evidence, and it is only through the latter and the principle of memorial prima facie credibility that empirical generalizations and the beliefs they support can be inductively supported by premises about past experience as well as about directly apprehended present experience. Further the credibility of our recollections, together with the whole range of empirical beliefs more or less dependent on them, can be solidified and increased through the mutual support or congruence of the whole, or can be diminished through incongruence.

A set of beliefs is congruent for Lewis (1946, 338) when the antecedent probability of each is increased by the assumption of the truth of the rest. A physical object statement P and the set of sensory conditionals that constitute its content form a congruent set, as we have seen. Indeed, by virtue of exhausting the empirical content of P, Lewis (1946, 348-9, footnote 6) thinks the sensory conditionals constitute a congruent set by themselves. In any case, the degree of warrant for the various elements of a mature system of empirical belief, especially one that counts as knowledge , depends on the inferential support the elements provide each other and the total empirical and memorial data present. However, there will be particular linkages of a priori probability relationships that would confer some degree of initial probability not just on what we recall but on simple generalizations from past experience and the expectations of future experiences and thus interpretations of experience these recollections inductively support, even in the hypothetical absence of (other) objective beliefs with which they are congruent or incongruent. Moreover, the recollections that support them and form part of an overall congruent system must have some degree of credibility independent of each other and the rest. The improbability of independently probable congruent recollections all being true were the beliefs they inferentially support and that inferentially support them not true makes it unlikely they are illusions of memory and increases the antecedent probability of the recollections and what they support (Lewis 1946, 352-3). Spelling out the antecedent and independent probability constraints is tricky.

The principle of the prima facie credibility of mnemic presentation of past experience can't itself be justified inductively for Lewis, on pain of circularity. Nor did he think it is simply a postulate — something we have to assume for empirical knowledge to be possible. Instead, he argued that it is constitutive of the lived world of experience and something for which there is no meaningful alternative. Sceptical alternatives designed to undermine the principle are ones that are inaccessible to knowledge and thus ones for which there is no criterion in experience. So it is an "analytic statement" that the past is knowable, and a similar claim was made for the relevance of past experience to the future, and thus for the knowability of empirical reality. The philosophical problem for Lewis (1946, 360-2) is to formulate correctly the criteria that "delimit empirical reality and explicate our sense of it".

In MWO, he defended induction in more detail by arguing that not every prediction is compatible with an evidence base, and that successive revision of one's predictions in the light of new experience can't help but make for more successful predictions (Lewis 1929, 367, 386). Nelson Goodman's well known "grue" example (Goodman 1955) poses problems for the relevance of the first claim and the force of the second. At other times, Lewis simply followed Hans Reichenbach in claiming that we can be assured only that if any procedures will achieve success in prediction, inductive ones will, without clearly distinguishing that claim from any attempt at an analytic justification of induction.

Rationally credible or warranted or justified belief, Lewis thought, is probable on the evidence, but the presentation of his views on probability was underdeveloped in MWO, and complex, and sometimes confusing in AKV. In AKV, Lewis defended an a priori account of probability or what he sometimes called "expectation". However, he rejected the Principle of Indifference often associated with a priori accounts, understood as the principle that in the absence of any reason for thinking one a priori possibility more likely than another, they are equiprobable (Lewis 1946, 306-314). The expectation or probability a/b of a proposition P is always relative to some set of empirical data or premises D. The expectation corresponds to an a priori valid estimate of the frequency of some property mentioned in P in some reference class mentioned in P , which estimate is derived from data or premises D , given the a priori valid principles of probabilistic inference, including the principles of induction. Hypothetical or conditional probability statements that are a priori valid license valid probabilistic inferences from premises about evidence or data to probabilistic conclusions. However, for Lewis, both hypothetical and categorical probability are always relativized to an evidence base, despite his occasional apparent talk of a priori valid probability statements as licensing inferences from empirical evidence to a (detachable) conclusion “Probably, P”.

Lewis rejected the view that probabilities are empirically based estimates of the limiting value of the frequency of instances of a property in a population, and thus expressed in non-terminating judgments. First, he thought that any attempt to define probabilities as the ratio of instances of one property among instances of another property as the latter approaches infinity would make probability judgments empirically untestable. Second, he argued that, if probability judgements were empirical frequency claims, then the probability judgements would themselves only be probable, something that can't be coherently accounted for. Nonetheless, Lewis recognized the need to assure ourselves rationally that the frequency as validly estimated from the data is closely in accord with the actual frequency and that there is nothing in the case at hand affecting the occurrence of the property which isn't taken into account in the specification of the reference class. Lewis dubbed this the “ reliability” of the determination of probability or expectation. He thought that reliability is a function of the adequacy of data (e.g., size of sample), the uniformity with which the frequency of some property in the data as a whole also holds for subsets of the data, and the proximateness or degree of resemblance between the data and the case at hand in P, all of which he also thought are logical relations.

So, in AKV, Lewis (1946, 305) claimed that the full statement of a probability judgement should be of the form “That c, having property F, will also have property G, is credible on data D, with expectation a/b and reliability R”, and is assertable in whatever sense D is. The judgement is valid when, in accordance with the a priori rules of probability and the correct rules of judging reliability, D gives the estimate a/b of the frequency of Fs among Gs, and D's adequacy, uniformity, and proximateness to the case in point, yields reliability R. A valid probability judgement is true when D is true, and is a categorical rather than hypothetical judgement when D is categorically asserted as true. Nonetheless, the assertion of the empirical data D is the only empirical element in the probability judgement, which otherwise has no testable implications for experience. However, the belief P, that c which has F is also G, is an empirical belief that may be rationally credible, empirically justified and warranted, in so far as D is given and the degree of assurance or belief corresponds to an a priori degree of probability (expectation) of P on D that is sufficiently reliable. Further, acceptance of P counts as empirical knowledge in so far as, firstly, P is true, secondly, the degree of probability or expectation of P on D is sufficiently high as to approach practical certainty, and, thirdly, D consists of all relevant data (Lewis, 1946, 314–15).

It is important to distinguish counterfactual statements of the form (S & A) → (h)E from a priori probability statements of the form “ Prob (E, on S and A) > .5”. Both express conditional probabilities. However, the former express ‘real’ connections knowable by induction from past experience. They constitute the analytically entailed consequences of an objective material objective statement P, but can't themselves be analytic truths. The latter, on the other hand, if true, are analytically true, knowable a priori, with zero intension, and entailed by any statement whatsoever, and so can hardly constitute the empirical meaning of contingently true material object statements. Yet, apart from denying that “→” can be understood either as material implication or strict implication, Lewis had little to say in print about what the truth conditions of subjunctive or counterfactual conditionals are. (Murphey (2005, 332) quotes correspondence from Lewis complaining that

Goodman and Chisholm in their writings miss the obvious interpretation of "If A were the case, then B would be the case", namely that A plus other premises of the (actual or hypothetical) case inductively justify the conclusion B. The problem is to interpret the remark so as to avoid turning counterfactuals into analytic truths.) Nonetheless, Lewis emphasized their importance, and the importance of the real connections they express, for the possibility of realism about the material world and the rejection of any sort of idealism or view that physical objects are simply mind-dependent collections of experiences (Lewis 1955, in Lewis 1970). The sensory conditionals (S & A) → E and (S & A) → (h)E can be true, as can the material object statement P that entails them, quite apart from the truth of the expressive statements S and A, or indeed, the presence of any empirical data warranting their assertion.

Chisholm (1948) raised the most important challenge to Lewis' claim that a physical object statement P can entail a set of counterfactual statements expressing claims about what experiences one would have were one to (seem to) carry out certain tests upon being presented with certain sensory cues. If P entails T, then for any Q consistent with P, P and Q also entail T. However, Chisholm argued, for any material object statement P and for any sensory conditional (S & A) → (h)E, there will be some other material object statement M about the circumstances of perception that is consistent with P, such that P and M can both be true while (S & A) → (h)E is false. For example, suppose P is "There is a doorknob before one" and (S & A) → (h)E is "If one were to seem to see a doorknob and have the experience of reaching out with one's hand, then, in all likelihood, one would seem to feel something hard and round", and M is "One's fingertips have been permanently anaesthetized". (Expanding the understanding of S to include sensory correlates of circumstances of perception, as Lewis (1946, 245-6) suggests, presumably just requires expanding the understanding of M with a little imagination.) A material object statement like P, therefore, doesn't entail sensory conditionals like (S & A) → (h)E . Instead of Lewis' empiricism about the meaning and justification of material object statements, Chisholm proposed that our spontaneous perceptual beliefs about the world, e.g., that one is seeing a doorknob, are prima facie justified just by virtue of being such spontaneous perceptual beliefs, quite apart from any inductive justification from present and past experience that might be reconstructed. Lewis' own defence of the prima facie credibility of memory, Chisholm thought, prepared the way for his alternative. Quine (1969), on the other hand, thought that Chisholm's problem just shows that what have consequences for experience and are tested by experience are never individual material object statements in isolation from each other but only sets of them or theories. Quine saved empiricism by drawing a holistic moral from the sort of problem Chisholm posed.

In a rare reply to critics, Lewis (1948) responded that Chisholm had misunderstood the implication of the probability qualifier. The familiar rule "If P entails T, then for any Q, P and Q entail T" doesn't apply when T is any kind of probability statement. E being improbable on P and M and S and A is perfectly consistent with E being probable on P and S and A, and so presumably doesn't undermine the claim that (S & A) → (h)E. However, this leaves the character of Lewis' empiricism puzzling. If the relative probability statements in question are the subjunctive conditionals, "(P and S and A) →(h)E" and "it is false that ((P and S and A and M) → (h)E)", then the statements in question are empirical propositions justified by induction. The justification for them thus will presuppose prior knowledge of the truth of material object statements like P and M, perhaps in the way Chisholm suggests, rather than explain how we can know such propositions solely on the basis of present and past experience of the given. On the other hand, if the relative probability statements are supposed to be a priori analytic statements, then it is the total set of such statements that constitutes the empirical meaning of P, statements like "Prob (E, given P and S and A and M) < .5" as much as "Prob (E, given P and S and A) > .5". Even when the relativization to other background material object statements isn't explicit, the probability statement would seem to be implicitly relative to some background assumption of material normality. In other words, Lewis would have to abandon his reductionism and agree with Quine's holistic conclusion that individual material object statements like P have "no fund of experiential implications to call their own" (Quine 1969, 79).

As noted earlier, Lewis (1929, 286) and Lewis (1936b) flirts with Quine's alternative. However, in AKV and in his reply to Chisholm, Lewis clearly repudiates it: if Chisholm were right that "no statement of objective fact had consequences for direct experience without further premises specifying objective conditions of perception", then, Lewis says, "the type of empiricism of which my account is one variant—verification-theories and confirmation-theories—will be altogether indefensible" (Lewis 1948, reprinted in Lewis 1970, 318). The result, he adds, would be a fatally flawed "coherence theory of empirical truth" that leaves us with "nothing..but skepticism".

14.7 THE GIVEN

Lewis' views about the given are at once among his best known and among his most criticized. The pre-analytic data for philosophical reflection is our "thick" experience and knowledge of the world around us, but reflection on this experience and knowledge reveals two elements: the given or immediate data of experience and the activity of thought whereby we conceptually interpret the given. The given in sense experience consists of specific sensuous qualities that we are immediately aware of when, for example, we take ourselves to be seeing or hearing or tasting or smelling or touching something, or even to be hallucinating or dreaming instead. These distinct qualities or qualia (singular quale) are the repeatable felt characters of experience, and include the felt goodness or felt badness of particular experiences or stretches of experience, as well as qualities of sight, sound, taste, smell, touch, motion, and other familiar modes of experience. On the other hand, the repeatability of these qualities, or the similarity of current instances to past instances, isn't something that is given to us. When we conceptually interpret the given, we form hypothetical expectations and make predictions, in the light of past experience, concerning experiences we would have were we to engage in specific actions, and so, in applying concepts, as Kant suggests, we relate our experiences to each other. However, usually, we do so automatically and without conscious reflection, in ways that express habitual attitudes and associations rather than engaging in (explicit) inference. The given, unlike our conceptual interpretation of it, isn't alterable by our will. It consists of what remains when we subtract from ordinary perceptual cognition all that could conceivably be mistaken (Lewis (1946), 182–3). Our apprehension of the given isn't, therefore, subject to any error and isn't subject to correction or verification or disconfirmation from further experience, and isn't, as a result, to be classified as knowledge. Any comparative classification of experience in terms of similarities and differences with other experiences, on the other hand, relates experiences to each other and and isn't certain. What we recall of past experience, even immediately, isn't given to us or certain, but, as he makes clear in AKV, our immediate recollection or sense of past experience as having been so and so is.

In MWO Lewis (1929, 401) says the given in experience never occurs in the absence of interpretation and characterizes the distinction as an "abstraction" of elements that are synthesized in our judgment, but which we may realize are common to quite different conceptualizations such as those of the adult and the child (Lewis 1929, 49-50). AKV is more circumspect. Although the given is what we are immediately aware of or directly apprehend as it guides and corrects our interpretations, it isn't something we focus on or attend to or are "clearly conscious"of in our automatic interpretations (Lewis 1946, 153) any more than in riding a bicycle we attend to or focus on the various sensory and motion and balance sensations that are elements in and guide our activity, though we could on reflection and perhaps did in learning (Lewis 1946, 10). In perceptual cognition, what is given in sense experience serves as a natural sign of future experience contingent on action in the light of past experience, and prompts the anticipation of such experience. What is given is of no interest to active beings apart from what it signifies for future experience and anticipations it prompts for action. (Lewis 1946, 10).

Nonetheless, he says "the validity of this interpretation is that and that only which could attach to it as an inductive inference from the given visual presentation...the incorrigible presentational element" (Lewis 1952a, reprinted in Lewis 1970, 326). What matters for the credibility or warrant or validity of the belief is that there is a logical, inferential relation between the belief and grounds in experience that prompt it in the light of past experience and that can be elicited on critical reflection and the character of the relation made clear enough for our theoretical purposes. However, the credibility or validity of the belief isn't created by the reflective attempt to elicit sufficient grounds (Lewis 1946, 186, 189, 262, 329-32) The given, thus, plays both a causal role as the ultimate or remote ground responsible for belief and an epistemic role as the ultimate justifying grounds of empirical belief. (Lewis 1946, 262, 328-30).

Probability, for Lewis, concerns a logical relation between a conclusion and premises, and a statement is categorically assertable with a degree of probability or credibility, or a belief warranted or credible to that degree, as opposed to being merely hypothetically probable a priori to that degree on premises, only to the extent that the premises or data are sufficiently credible or warranted or probable. (Lewis 1946, 315-27). Ultimately, the conclusion must be warranted or credible or probable on premises or data that are certain, not just true, and not just warranted or credible only on other premises or data, though we may never reach them ordinarily in showing probability or justification. Otherwise we have "an indefinite regress of the merely probable...and the probability will fail to be genuine" (Lewis 1946, 186). Here he echoes MWO where he says that the validity of a probability judgement is a relation between the judgement and "ultimate premises" that (a) "may verbally be quite remote", unlike the "immediate premises" we might initially and normally cite, that (b) must be a "certainty" rather than merely probable on yet further premises, and that (c) must be "actual given data for the individual" (Lewis 1929, 328-9). Lewis is defending a normative standard for empirical knowledge that he thinks is implicit in cognition and revealed on reflection but which is also psychologically and verbally remote from everyday cognitive practices of justifying beliefs to ourselves or others in the light of more proximate assumptions taken for granted in he context of inquiry or discussion. Some pragmatists might feel there is tension here.

In MWO he also famously says that the given is "ineffable" (Lewis 1929, 53). So how can what is ineffable even be true, and how can what is neither true nor false, nor as a result, neither probably true nor probably false, serve as the ultimate premises of a priori valid logical probability relations licensing belief or assertion with probability or credibility? And how can we anticipate or predict future experience that isn't yet given, except in conceptual or comparative terms that won't allow for decisive falsification? Again, there may seem to be tension in Lewis' views of the given and the epistemic role he assigns it. Lewis (1936a and 1936b, reprinted in Lewis 1970, 155-7, 292-3) clearly recognizes the logical and epistemological problems, and he responds by introducing categories of expressive statements and the expressive use of language. This carries over into AKV. Expressive statements like “It seems as though I am seeing a red round thing” serve to convey or express or denote what we directly apprehend in experience without conceptualizing and interpreting it. They are true by virtue of the qualitative character of experience they express and are verified by it, and false only when we knowingly lie about our experience, and the ineffability of what they express just consists in their not implying possibilities of further experience. Moreover, their truth is something that we know, or, as he more carefully and repeatedly says in AKV, something about which we are certain (Lewis 1946, 171-2, 183, 204, 327). The expressive use of language is to convey or express what is not only directly apprehended but what may be directly apprehensible in the future or, perhaps, was directly apprehensible in the past. (Lewis 1946, 179). Nonetheless, Lewis notes that the expressive use of language is needed only for the discussion of knowledge, not for knowledge itself (Lewis 1946, 183; 1952a, reprinted in Lewis 1970, 327). So perhaps it isn't surprising that later he also talks of “immediately given facts of sense” and “facts of our seeming to remember” (Lewis 1946, 327, 353) and “datum facts” and “logical relations of facts” (Lewis 1952a, reprinted in Lewis 1970, 325).

Sellars (1963, 132) thought the classical empiricist given was an inconsistent triad of three claims: (1) being appeared to as if there were something red entails non-inferentially knowing that one is appeared redly to, (2) the ability to be appeared to is unacquired, and (3) the ability to know facts of the form x is F is acquired. Lewis clearly denied (1), but he recognized this was the result of a choice about how to use “know” and that others, “without fault”, might, choose to extend it to direct apprehensions of sense because they are certain for us (Lewis 1946, 183). However, if Lewis followed suit and granted (1), it seems he would deny (3) on the grounds that the ability to be certain of the given wasn't acquired. Sellars might question whether such certainty, or the expressive language in which it was couched, was sufficiently rich in content to support other knowledge inferentially. In any case, Lewis' defense of the certainty of the given rests on two claims. First, it is just an undeniable fact, apparent to anyone who reflects on experience, that there is a sensuous character of experience that we are aware of and can't be mistaken about and that, until it fades to memory, isn't subject to correction and isn't further confirmable. As Lewis (1952a, reprinted in Lewis 1970, 329) put it in his symposium on the given with Reichenbach (1952) and Goodman (1952), there is no requirement of "inductive consistency" on protocols or expressive statments. Second, the supposition that probability is always relative to something else that is itself only probable means that probabilities can never get off the ground. As Lewis famously says, "if anything is to be probable, something must be certain" (Lewis 1946, 186). Goodman (1952), in his symposium contribution, argued that the premises relative to which other statements are credible or probable just have to be initially credible on their own to some degree, not certain, though subject to future confirmation or disconfirmation in the light of further experience. So long as they were initially credible on their own rather than because something else was initially credible, Lewis' regress failed. This is a view that attracted many epistemologists after Lewis in some form or other.

Lewis' response is instructive for his understanding of epistemology. For Lewis (1952a, reprinted in Lewis 1970, 330), a principal task of epistemology is with the "validity" of knowledge, that is to say with the justification or warrant for cognition that distinguishes empirically warranted belief from lucky or unlucky guess or hazard of belief. If a class of beliefs in principle may be false, we need some reason or grounds for thinking its members true or likely to be true, especially if we plan to base other beliefs on them. That requires present or past justifying grounds of belief, not just future potential for verification or confirmation as he thinks Goodman proposes. Otherwise, we confuse justification with verification, or ignore the former for the latter. Nor can the grounds consist solely in other beliefs that might be mistaken without grounds for thinking them true or likely to be true, or in beliefs that stand in conditional probabilistic relations to each other, as he thinks Reichenbach proposes, without any antecedent probabilities deriving from something else (Lewis 1952a, reprinted in Lewis 1970, 328). Lewis acknowledges that his (somewhat traditional) concerns with validation or justification, skepticism, and the need for given justifying grounds, leads him to depart from or supplement traditional pragmatic theories.

Finally, we can't directly verify the existence of other subjects of experience or what is given to them in their experience. Nonetheless, Lewis (1934, 1941b) claimed that by empathy, in terms of our own conscious experience, we can imagine or envisage the conscious experience of others, rather than simply our own experience of others and their bodies and our interactions with them. Moreover, the supposition of another consciousness like ours, with a body like ours, can be indirectly confirmed and supported by induction. However, Lewis provided no details concerning this inductive support for our belief in other minds.

14.8 ACTION, THE GOOD, AND THE RIGHT

In contrast with those logical positivists who thought that statements of value merely express attitudes, pro or con, to objects, persons, or situations, but are neither true nor false, Lewis (1946, 396–98) thought that statements of value were as true or false as other empirical statements, and every bit as empirically verifiable or falsifiable, confirmable or disconfirmable. True, felt value qualia, felt goodness and badness, are given to us and directly apprehended in experience or stretches of experience, and “expressive” statements must be used to indicate or convey them. However, such statements, like Lewis’ other “expressive” statements, may be true or false (see previous section), and simply convey the occurrence of given qualia in experience and no more, instead of indicating the existence of objects, situations, or persons, and expressing our attitudes to them. Moreover, there are also for Lewis terminating judgments of value concerning what the felt value of some experiences indicates about the felt value of further experiences. Finally, there are “objective” judgements of value: judgements attributing value to persons, objects, and objective situations, in so far as they have the potential, depending on circumstances, to produce felt goodness or badness in us or others. These are non-terminating judgements of value and are empirically confirmable or disconfirmable by induction just like any other objective empirical judgement. Lewis thus claims that his theory of value is thoroughly naturalistic and humanistic, rather than transcendental, but still objectivist.

The felt goodness of experience is what is intrinsically good or valuable for its own sake. It is only experience in so far as it has such value quality that is intrinsically good rather than merely extrinsically valuable for its contribution to something else that is intrinsically valuable. Value and disvalue are modes or aspects of experience to which desire and aversion are “addressed” (Lewis 1946, 403). Lewis denies that “pleasure” is adequate to the wide variety of what is found directly good in experience, and thus thinks it inadequate as a synonym for “good”. However, as Frankena (1964) argues, for Lewis directly found goodness still seems to be as natural a quality or property of certain experiences as any other qualia directly apprehended in experience. Nonetheless, the value of a stretch of experience, indeed a whole life, isn't just the value (and disvalue) of the parts, and in AKV, Lewis criticized Bentham’s attempt at a calculus of values. For Lewis, the intrinsic value found in the experience of a symphony isn't just the sum of the intrinsic value of the movements taken individually, but reflects the character of the symphony as a temporal Gestalt. What is ultimately good for Lewis is the quality of a life found good in the living of it. (Lewis 1952b in Lewis 1970, 179) The constituent experiences thus might have value for their own sake, but also value for their contribution to the value of the whole life of which they are parts.

However, Lewis thought that judgements about how a valued experience contributes to the value of a life as a whole, unlike some terminating judgements about how one valued experience will yield another valued experience, are not decisively verifiable or falsifiable. First, any attempt to apprehend a life as a whole and the value of it as experienced goes beyond the specious present of experience and relies on memory and expectation of past and future experience and their values, and thereby leaves room for error. Second, any attempt to simplify the problem by breaking a whole life into parts and apprehending their value, and then calculating the probability of their contributing to a good life as a whole, also leaves room for error.

The value of an object consists in its potentiality for conducing to intrinsically valuable experiences, and is thus a real connection between objects, persons, and the character of experience, which we can be empirically warranted in accepting on the basis of the empirical evidence and the probability on the evidence of such objects yielding such intrinsically valuable experiences. For Lewis (1946, 432), therefore, no object has intrinsic value. Nonetheless, objects can have inherent value in so far as the good which they produce is disclosable in the presence or observation of the object itself rather than some other object. Lewis (1946, Ch. 14) contrasted aesthetic value with cognitive and moral value, not by virtue of distinctive characters of their felt goods, but chiefly by distinctive attitudes to experience. The aesthetic attitude is one of disinterested interest in the presented, attentiveness to the given in its own right, as opposed to the cognitive attitude's concern with prediction and significance for future experience, and the concern on the part of the attitude of action or morality with the pursuit of absent but attainable goods. Thanks to these differences, aesthetic values in experience tend to be of high degree and long lasting and don't require exclusive possession, and aesthetic values in objects are inherent ones.

Lewis recognized that potentialities are in various ways relative to particular circumstances and manners of observation. There is thus a plurality of judgements of the value of objects, of the various ways in which they can contribute and fail to contribute to intrinsically valuable experiences, and an apparent contradictory nature to incomplete verbal statements of them (e.g., “X is good”, “X isn't good”). For Lewis (1946, 528), issues about the relativity or subjectivity of judgements of the value of objects aren't issues about the empirical truth of attributions of value to objects, but just issues about whether the conditions under which an object produces directly apprehended value are peculiar to the nature and capacities of a particular person and thus not indicative of the possibility of similar value finding on the part of other persons. Quine (1981) argued that variation within and among individuals and societies, and the variable and open ended character of what they find valuable, means that predicates like “pleases” or “ feels good” don't support inductive inferences from case to case in the way that “green” or “conducts electricity” do. Skepticism concerning the prospects for empirical content and empirical truth of attributions of value to objects is thus in order. Lewis (1946, 323), on the other hand, seems to have thought that this contention implies that no one could ever act with empirical warrant to improve his own lot in life or do any others good, an absurdity in his view. Lewis argued at length for the possibility of empirically warranted judgements of the social or impersonal value of objects. The key is that “value to more than one person is to be assessed as if their several experiences were to be included in that of a single person” (Lewis (1946, 550). Rawls (1971, 188–90) criticized Lewis for mistaking impersonality for impartiality, and denied the relevance of Lewis' account of impersonal value for questions of justice, at least, for which impartiality is key.

An action, for Lewis (1955, 49), is subjectively right, and one we are not to be blamed for doing, if we think it objectively right. An action is objectively right if it is correctly judged on the evidence that its consequences are such as it will be right to bring about. That requires that their pursuit violates no categorical rational imperative or principle.

Check Your Progress 1

Note: a) Use the space provided for your answer.

b) Check your answers with those provided at the end of the unit.

1. What do you know the Brief Biography Lewis?

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2. Discuss the overview of Conceptual Pragmatism.

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3. Highlight Logic and Language.

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4. Discuss the A Priori and the Analytic.

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14.9 LET US SUM UP

Lewis (1952b, 1952c, 1955, Chapter 5) outlines categorical rational imperatives of doing and thinking, or versions of one rational imperative, in various ways, formulations, and detail. The general idea is laid out briefly in AKV (Lewis (1946, 480–82). To be subject to imperatives is to find a constraint of action or thought in what is not immediate. To be rational is to be capable of constraint by prevision of some future good or ill, and subjection to imperatives is simply a feature of living in human terms. Rationality turns on consistency, and the logical is derivative from the rational. Indeed consistency of thought is for the sake of and aimed at consistency in action, which in turn derives from consistency in willing, i.e., of purposing and setting a value on. Logical consistency turns on nowhere repudiating that to which we anywhere commit ourselves to in our thought, and consistency in general consists in not accepting now what we are unwilling to commit to elsewhere or later. Consistency in what we think and do requires and is required by conformity to principles.

So there is a categorical rational imperative of consistency, " Be consistent in valuation and in thought and action" (Lewis 1946, 481) the basis of which is simply a datum of human nature, and a broader imperative of cogency or basing one's beliefs on cogent reasoning from evidence (Lewis 1952b, 1952c), an imperative of prudence, "Be concerned about yourself in future and on the whole", and an imperative of justice, “No rule of action is right except one which is right in all instances and therefore right for everyone” (Lewis 1946, 481–2). These principles are simply a priori explications of the rational or moral sense possessed by most humans. Certainly, this might be challenged. In any case, Lewis thinks that where that sense is lacking, argument for the principles is pointless, and he concludes AKV by claiming that “valuation is always a matter of empirical knowledge” but “what is right and what is just can never be determined by empirical facts alone” (Lewis 1946, 554).

The problem remains of reconciling the imperatives of prudence and (social) justice in practice, of reconciling the good for oneself with the good for others in our self-directed, principled, thinking and doing. What aids us is that, through language and civilization, humans remember as a species and not merely as individuals. What we are justified in thinking thereby is that human achievement and social progress require autonomous, self-criticizing and self-governing individuals, and that individual achievement and realization of cherished goods requires membership in a social order of individuals co-operating in the pursuit of values cherished in common. The contrast between individual prudence and social justice seems fundamental, Lewis concludes, perhaps rather optimistically, only by forgetting this (Lewis 1952b).

14.10 KEY WORDS

Empirical: Empirical evidence is the information received by means of the senses, particularly by observation and documentation of patterns and behavior through experimentation. The term comes from the Greek word for experience.

Self-governing: Self-governance, self-government, or self-rule is the ability of a group or individual to exercise all necessary functions of regulation without intervention from an external authority

14.11 QUESTIONS FOR REVIEW

1. What do you know about the Empirical Knowledge?

2. Discuss the concept of The Given.
3. What do you understand the Action, the Good, and the Right?

14.12 SUGGESTED READINGS AND REFERENCES

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14.13 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Notes

1. See Section 14.2
2. See Section 14.3
3. See Section 14.4
4. See Section 14.5